Math 101 Fall 2000 Exam 1

Instructor: Richard Stong

Tuesday, October 3, 2000

Instructions: This is a closed book, closed notes exam. Use of calculators is not permitted. You have **one hour and fifteen minutes**. Do all 8 problems. Please do all your work on the paper provided.
Please print you name clearly here.

Print name: ______________________________

Upon finishing please sign the pledge below:
On my honor I have neither given nor received any aid on this exam.

____

Grader's use only:

1. ________ /15

2. ________ /10

3. ________ /10

4. ________ /10

5. ________ /20

6. ________ /15

7. ________ /10

8. ________ /10

1. [15 points] Find the following limits, if they exist.

(a) $\lim_{x\to -2} \frac{x^2-x-6}{x+2}$

$$\lim_{x\to -2} \frac{x^2-x-6}{x+2} = \lim_{x\to -2} \frac{(x+2)(x-3)}{x+2} = \lim_{x\to -2} x-3 = -2-3 = -5.$$

(b) $\lim_{x\to 0} \frac{\tan 5x}{x}$

$$\lim_{x\to 0} \frac{\tan 5x}{x} = \lim_{x\to 0} \frac{\sin 5x}{5x} \cdot \frac{5}{\cos 5x} = \lim_{x\to 0} \frac{\sin 5x}{5x} \cdot \lim_{x\to 0} \frac{5}{\cos 5x} = 1 \cdot \frac{5}{1} = 5.$$

2. [10 points] Let f be the function defined by

$$f(x) = \begin{cases} 3 - x & \text{if } x < 2 \\ 0 & \text{if } x = 2 \\ 2x^2 - 7 & \text{if } x > 2 \end{cases}$$

Find $\lim_{x \to 2^+} f(x)$, $\lim_{x \to 2^-} f(x)$, and $\lim_{x \to 2} f(x)$ (if they exist). Is f continuous at $x = 2$?

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} 2x^2 - 7 = 8 - 7 = 1.$$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} 3 - x = 3 - 2 = 1.$$

Since the two one-sided limits agree we also have

$$\lim_{x \to 2} f(x) = 1$$

However $f(2) = 0 \neq 1$, so f is not continuous at $x = 2$.

3. [10 points] (a) Give the formal, mathematical definition of the derivative of a function f.

(b) Find the derivative of $f(x) = \frac{1}{x+2}$ **using the definition of the derivative**. (No credit will be given for finding the derivative by other means.)

(a) **Definition.**

$$\frac{df}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

(b)

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{\frac{1}{x+h+2} - \frac{1}{x+2}}{h} \\
&= \lim_{h \to 0} \frac{(x+2) - (x+h+2)}{h(x+2)(x+h+2)} \\
&= \lim_{h \to 0} \frac{-h}{h(x+2)(x+h+2)} \\
&= \lim_{h \to 0} \frac{-1}{(x+2)(x+h+2)} \\
&= \frac{-1}{(x+2)^2}.
\end{aligned}$$

4. [10 points] Find the equation of the tangent line to the graph of $y = \sqrt[3]{x}$ at $x = 8$.

$$\frac{dy}{dx} = \frac{d(x^{1/3})}{dx} = \frac{1}{3}x^{-2/3}.$$

So at $x = 8$ we have

$$\left.\frac{dy}{dx}\right|_{x=8} = \frac{1}{3}8^{-2/3} = \frac{1}{12}.$$

Therefore the tangent line has slope $1/12$ and passes through the point $(8, \sqrt[3]{8}) = (8, 2)$. Hence it is

$$y - 2 = \frac{1}{12}(x - 8) \text{ or } y = \frac{1}{12}x + \frac{4}{3}.$$

5. [20 points] Find the derivatives of the following functions.

(a) $f(x) = 1 + 3\sqrt{x} + 2x^2 - 6x^{-3}$

Since $f(x) = 1 + 3x^{1/2} + 2x^2 - 6x^{-3}$, we have

$$f'(x) = \frac{3}{2}x^{-1/2} + 4x + 18x^{-4}.$$

(b) $g(t) = \frac{e^{2t+1}}{1+3t^2}$

$$\begin{aligned} g'(t) &= \frac{(1+3t^2)\frac{d(e^{2t+1})}{dt} - e^{2t+1}\frac{d(1+3t^2)}{dt}}{(1+3t^2)^2} \\ &= \frac{(1+3t^2)e^{2t+1}\frac{d(2t+1)}{dt} - e^{2t+1}\cdot 6t}{(1+3t^2)^2} \\ &= \frac{2(1+3t^2)e^{2t+1} - e^{2t+1}\cdot 6t}{(1+3t^2)^2}. \end{aligned}$$

(c) $F(t) = \sqrt{t}\sin(t^4)$

$$\begin{aligned} F'(t) &= \frac{d(t^{1/2})}{dt}\sin(t^4) + \sqrt{t}\frac{d(\sin(t^4))}{dt} \\ &= \frac{1}{2}t^{-1/2}\sin(t^4) + \sqrt{t}\cos(t^4)\frac{d(t^4)}{dt} \\ &= \frac{1}{2}t^{-1/2}\sin(t^4) + 4t^3\sqrt{t}\cos(t^4). \end{aligned}$$

(d) $f(x) = \left(2\ln(2 + 3x^{-2}) + 7\right)^8$

$$\begin{aligned} f'(x) &= 8\left(2\ln(2+3x^{-2})+7\right)^7 \frac{d}{dx}\left(2\ln(2+3x^{-2})+7\right) \\ &= 16\left(2\ln(2+3x^{-2})+7\right)^7 \frac{d}{dx}\ln(2+3x^{-2}) \\ &= 16\left(2\ln(2+3x^{-2})+7\right)^7 \frac{1}{2+3x^{-2}}\frac{d(2+3x^{-2})}{dx} \\ &= 16\left(2\ln(2+3x^{-2})+7\right)^7 \frac{1}{2+3x^{-2}}\cdot(-6x^{-3}) \\ &= -96x^{-3}\left(2\ln(2+3x^{-2})+7\right)^7 \frac{1}{2+3x^{-2}} \end{aligned}$$

6. [15 points] A sector is removed from a circular piece of cardboard of radius 10 cm. The remaining cardboard is folded so the ends of the sector join to form a cone. What is the maximum possible volume of the resulting cone? (The volume of a cone with height h and radius of the base r, is $V = \frac{1}{3}\pi r^2 h$.)

Let h be the height of the cone and r its radius (in cm). Looking at a the cone from the side, we see that there is a right triangle with height h (the axis of the cone) and base r (a radius of the cone) and hypotenuse 10 (a radius of the original cardboard). Thus $r^2 + h^2 = 100$. Using this to eliminate r from the volume of the cone we see the volume is

$$V(h) = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(100 - h^2)h = \frac{1}{3}\pi(100h - h^3).$$

Since h and $r^2 = 100 - h^2$ must both be non-negative, we see $0 \leq h \leq 10$ is the domain. We compute

$$V'(h) = \frac{1}{3}\pi(100 - 3h^2).$$

This is zero only for $h = \pm 10/\sqrt{3}$, but the negative is outside the domain. Since $V(0) = 0$ and $V(10) = 0$, the maximum possible volume is

$$V(10/\sqrt{3}) = \frac{1}{3}\pi\left(100 - \left(\frac{10}{\sqrt{3}}\right)^2\right)\frac{10}{\sqrt{3}} = \frac{1}{3}\pi\left(100 - \frac{100}{3}\right)\frac{10}{\sqrt{3}} = \frac{2000\pi}{9\sqrt{3}}.$$

7. [10 points] Use implicit differentiation to find $\frac{dy}{dx}$ if $x \tan y = \cos(x + y)$.

Implicit differentiation gives

$$\begin{aligned} \frac{dx}{dx}\tan y + x\frac{d\tan y}{dx} &= \frac{d\cos(x+y)}{dx} \\ \tan y + x\sec^2 y\frac{dy}{dx} &= -\sin(x+y)\frac{d(x+y)}{dx} = -\sin(x+y)\left(1 + \frac{dy}{dx}\right). \end{aligned}$$

Rearranging gives

$$\tan y + \sin(x+y) + \big(x\sec^2 y + \sin(x+y)\big)\frac{dy}{dx} = 0.$$

Solving for dy/dx gives

$$\frac{dy}{dx} = -\frac{\tan y + \sin(x+y)}{x\sec^2 y + \sin(x+y)}.$$

8. [10 points] A baseball diamond is a square with side length 90 ft. A batter hits the ball and runs towards first base with a velocity of 24 ft/sec. At what rate is his distance from 3rd base increasing when he is halfway to first base?

Let x be the distance in feet from the runner to home plate and D the distance in feet from the runner to third base. Home plate, third base and the runner are at the vertices of a right triangle with sides 90 (third to home) and x (home to runner) and hypotenuse D (runner to third). Hence $90^2 + x^2 = D^2$. We are told

$$\frac{dx}{dt} = 24$$

and asked for

$$\left.\frac{dD}{dt}\right|_{x=45}.$$

Differentiating the equation relating x and D with respect to t gives

$$2x\frac{dx}{dt} = 2D\frac{dD}{dt} \quad \text{or} \quad \frac{dD}{dt} = \frac{x}{D}\frac{dx}{dt}.$$

When $x = 45$, we compute $D^2 = 90^2 + 45^2 = 45^2(4+1)$ so $D = 45\sqrt{5}$. Thus

$$\left.\frac{dD}{dt}\right|_{x=45} = \frac{45}{45\sqrt{5}} \cdot 24 = \frac{24}{\sqrt{5}}.$$

Thus the distance from the runner to third base is increasing at $24/\sqrt{5}$ feet per second when he is halfway to first.