Math 101 Fall 2001 Exam 1 **Solutions**

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Instructions: This is a closed book, closed notes exam. Use of calculators is not permitted. You have **one hour and fifteen minutes**. Do all 7 problems. Please do all your work on the paper provided. You must show your work to receive full credit on a problem. An answer with no supporting work will receive no credit.
Please print you name clearly here.

Print name: ______________________________

Upon finishing please sign the pledge below:
On my honor I have neither given nor received any aid on this exam.

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Grader's use only:

1. ________ /15

2. ________ /10

3. ________ /10

4. ________ /20

5. ________ /10

6. ________ /20

7. ________ /15

1. [15 points] Find the following limits, if they exist.

   (a) $\lim\limits_{x\to 2} \frac{x^2+x-6}{x^2-3x+2}$

$$\lim_{x\to 2} \frac{x^2+x-6}{x^2-3x+2} = \lim_{x\to 2} \frac{(x-2)(x+3)}{(x-2)(x-1)} = \lim_{x\to 2} \frac{x+3}{x-1} = \frac{2+3}{2-1} = 5.$$

   (b) $\lim\limits_{\theta\to 0} \frac{\sin 2\theta}{\tan 3\theta}$

$$\lim_{\theta\to 0} \frac{\sin 2\theta}{\tan 3\theta} = \lim_{\theta\to 0} \frac{\sin 2\theta \cos 3\theta}{\sin 3\theta} = \lim_{\theta\to 0} \frac{\sin 2\theta}{2\theta} \cdot \frac{3\theta}{\sin 3\theta} \cdot \frac{2\cos 3\theta}{3}$$

$$= \frac{\lim\limits_{\theta\to 0} \frac{\sin 2\theta}{2\theta} \cdot \lim\limits_{\theta\to 0} \frac{2\cos 3\theta}{3}}{\lim\limits_{\theta\to 0} \frac{\sin 3\theta}{3\theta}} = \frac{1 \cdot \frac{2}{3}}{1} = \frac{2}{3}.$$

   (c) $\lim\limits_{x\to 4} \frac{\sqrt{x}-2}{x-4}$

$$\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4} = \lim_{x\to 4} \frac{\sqrt{x}-2}{(\sqrt{x}-2)(\sqrt{x}+2)} = \lim_{x\to 4} \frac{1}{\sqrt{x}+2} = \frac{1}{4}$$

or

$$\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4} = \lim_{x\to 4} \frac{(\sqrt{x}-2)(\sqrt{x}+2)}{(x-4)(\sqrt{x}+2)} = \lim_{x\to 4} \frac{x-4}{(x-4)(\sqrt{x}+2)}$$

$$= \lim_{x\to 4} \frac{1}{\sqrt{x}+2} = \frac{1}{4}.$$

2. [10 points] Let f be the function defined by

$$f(x) = \begin{cases} \frac{3-x}{|x-3|} & \text{if } x < 3 \\ -1 & \text{if } x = 3 \\ x^2 - 4x + 4 & \text{if } x > 3 \end{cases}$$

Find $\lim_{x\to 3^+} f(x)$, $\lim_{x\to 3^-} f(x)$, and $\lim_{x\to 3} f(x)$ (if they exist). Is f continuous at $x = 3$?

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} x^2 - 4x + 4 = 3^2 - 4\cdot 3 + 4 = 1.$$

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \frac{3-x}{|x-3|} = \lim_{x\to 3^-} \frac{3-x}{3-x} = \lim_{x\to 3^-} 1 = 1.$$

Since the left and right hand limits agree $\lim_{x\to 3} f(x) = 1$ exists. However since $f(3) = -1$ does not agree with the limit f is not continuous at $x = 3$.

3. [10 points] Find the derivative of $f(x) = \frac{1}{3x-1}$ **using the definition of the derivative**. (No credit will be given for finding the derivative by other means.)

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{3x+3h-1} - \frac{1}{3x-1}}{h}$$

$$= \lim_{h\to 0} \frac{(3x-1) - (3x+3h-1)}{h(3x-1)(3x+3h-1)} = \lim_{h\to 0} \frac{-3h}{h(3x-1)(3x+3h-1)}$$

$$= \lim_{h\to 0} \frac{-3}{(3x-1)(3x+3h-1)} = \frac{-3}{(3x-1)^2}.$$

4. [20 points] Find the derivatives of the following functions.

(a) $f(x) = \frac{3x-7}{x^2+4x+1}$

$$\begin{aligned}\frac{df}{dx} &= \frac{\frac{d(3x-7)}{dx}(x^2+4x+1) - (3x-7)\frac{d(x^2+4x+1)}{dx}}{(x^2+4x+1)^2} \\ &= \frac{3(x^2+4x+1) - (3x-7)(2x+4)}{(x^2+4x+1)^2}.\end{aligned}$$

(b) $g(t) = (1+t^2)^3 \cdot \sqrt{2+t}$

$$\begin{aligned}\frac{dg}{dt} &= \frac{d((1+t^2)^3)}{dt}\sqrt{2+t} + (1+t^2)^3\frac{d\sqrt{2+t}}{dt} \\ &= 3(1+t^2)^2(2t)\sqrt{2+t} + (1+t^2)^3\frac{1}{2\sqrt{2+t}}.\end{aligned}$$

(c) $y(x)$ if $y = \frac{1}{2u^2} + u^2$ and $u = x^{1/3} - \frac{1}{3x}$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = (-u^{-3}+2u)\left(\frac{1}{3}x^{-2/3} + \frac{1}{3}x^{-2}\right)$$

$$= \left(-\left(x^{1/3} - \frac{1}{3x}\right)^{-3} + 2\left(x^{1/3} - \frac{1}{3x}\right)\right)\left(\frac{1}{3}x^{-2/3} + \frac{1}{3}x^{-2}\right).$$

(d) $F(x) = \left(3 - (2+5x^2)^6\right)^{-1/5}$

$$\begin{aligned}\frac{dF}{dx} &= -\frac{1}{5}\left(3-(2+5x^2)^6\right)^{-6/5}\frac{d(3-(2+5x^2)^6)}{dx} \\ &= \frac{6}{5}\left(3-(2+5x^2)^6\right)^{-6/5}(2+5x^2)^5\frac{d(2+5x^2)}{dx} \\ &= 12x\left(3-(2+5x^2)^6\right)^{-6/5}(2+5x^2)^5.\end{aligned}$$

5. [10 points] Find the equation of the tangent line to the graph of $y = 3x - \frac{1}{x}$ at the point $(1, 2)$.

Since
$$\frac{dy}{dx} = 3 + \frac{1}{x^2},$$
we see
$$\left.\frac{dy}{dx}\right|_{x=1} = 4.$$
Therefore the slope of the tangent line through $(1, 2)$ is 4. Thus the tangent line is given by $(y - 2) = 4(x - 1)$ or $y = 4x - 2$.

6. [20 points] Consider the function $f(x) = x\left(\frac{5}{2} - \frac{x}{2}\right)^{2/3}$ on $[1, 7]$. Find the maximum and minimum of f on this interval. Be sure to show all the steps you need to show in order to justify that your answers really are the maximum and minimum. (It may help to know that $f(1) = 2^{2/3} = 1.5874....$)

Since cube roots are always continuous f is continuous on $[1, 7]$. To find the critical points we compute

$$f'(x) = \left(\frac{5}{2} - \frac{x}{2}\right)^{2/3} - \frac{x}{3}\left(\frac{5}{2} - \frac{x}{2}\right)^{-1/3}.$$

The derivative is not defined at $x = 5$, therefore thatis a critical point. Setting $f'(x) = 0$ gives

$$\left(\frac{5}{2} - \frac{x}{2}\right)^{2/3} = \frac{x}{3}\left(\frac{5}{2} - \frac{x}{2}\right)^{-1/3}$$

or upon clearing denominators $3(5 - x) = 2x$, hence $x = 3$. Thus the maximum and minimum must occur either at one of the endpoints $x = 1$ and $x = 7$ or at one of the critical points $x = 3$ or $x = 5$. Since

$$f(1) = 2^{2/3} = 1.5874...$$
$$f(3) = 3 \cdot 1^{2/3} = 3$$
$$f(5) = 5 \cdot 0^{2/3} = 0$$
$$f(7) = 7 \cdot (-1)^{2/3} = 7$$

we see the maximum is 7 attained at $x = 7$ and the minimum is 0 attained at $x = 5$.

7. [15 points] A poster is to have an area of 600 in^2 with 1 inch margins at the bottom and sides and a 2 inch margin on top. What dimensions will give the largest printed area?

Let x be the width of the poster in inches and y the height of the poster in inches. The printed area will therefore be a rectangle with width $x-2$ and height $y-3$. Hence it will have area $A = (x-2)(y-3)$. Since the total area is 600, x and y are related by $xy = 600$, or $y = 600/x$. Thus $A(x) = (x-2)(600x^{-1} - 3) = 606 - 3x - 1200x^{-1}$. To allow for the margins, we must have $x \geq 2$ and $y \geq 3$ hence $x \leq 200$.

Thus we want to maximize $A(x) = 606 - 3x - 1200x^{-1}$ on $[2, 200]$. $A(x)$ is continuous except at $x = 0$, hence on $[2, 200]$. We compute $A'(x) = -3 + 1200x^{-2}$ which is always defined for $x \neq 0$. Solving $A'(x) = 0$ gives $3 = 1200x^{-2}$ or $x^2 = 400$ or $x = \pm 20$. Thus $x = 20$ is the only critical point in our interval. Since $A(2) = A(200) = 0$ and $A(20) = 486$, the maximal printed area is 486 in^2 which is attained if $x = 20$ and $y = 30$.