Math 101 Fall 2002 Exam 1 **Solutions**

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Instructions: This is a closed book, closed notes exam. Use of calculators is not permitted. You have **one hour and fifteen minutes**. Do all 7 problems. Please do all your work on the paper provided. You must show your work to receive full credit on a problem. An answer with no supporting work will receive no credit.
Please print you name clearly here.

Print name: ______________________________

Upon finishing please sign the pledge below:
On my honor I have neither given nor received any aid on this exam.

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Grader's use only:

1. ________ /15

2. ________ /15

3. ________ /10

4. ________ /20

5. ________ /10

6. ________ /15

7. ________ /15

1. [15 points] Evaluate the following limits, if they exist.

   (a) $\lim_{x\to -2} \frac{x^2+x-2}{x^2-x-6}$

   Solution: Plugging in $x = -2$ gives $0/0$, hence both the numerator and denominator have roots at $x = -2$ and a factor of $x + 2$. Factoring and cancelling gives

$$\lim_{x\to -2} \frac{x^2+x-2}{x^2-x-6} = \lim_{x\to -2} \frac{(x+2)(x-1)}{(x+2)(x-3)} = \lim_{x\to -2} \frac{(x-1)}{(x-3)} = \frac{3}{5}.$$

   (b) $\lim_{\theta\to 0} \frac{\tan 2\theta}{\sin 5\theta}$

   Solution: Rewriting the limit gives

$$\lim_{\theta\to 0} \frac{\tan 2\theta}{\sin 5\theta} = \lim_{\theta\to 0} \frac{\sin 2\theta}{\sin 5\theta \cos 2\theta} = \lim_{\theta\to 0} \frac{\sin 2\theta}{2\theta} \cdot \frac{5\theta}{\sin 5\theta} \cdot \frac{2}{5\cos 2\theta} = 1 \cdot 1 \cdot \frac{2}{5} = \frac{2}{5}.$$

   (c) $\lim_{x\to 1} \frac{\frac{2}{x+1}-1}{x-1}$

   Solution: Multiplying top and bottom by $x + 1$ to rewrite the limit as a rational function gives

$$\lim_{x\to 1} \frac{\frac{2}{x+1}-1}{x-1} = \lim_{x\to 1} \frac{2-(x+1)}{(x-1)(x+1)} = \lim_{x\to 1} \frac{1-x}{(x-1)(x+1)} = \lim_{x\to 1} \frac{-1}{x+1}$$
$$= -\frac{1}{2}.$$

2. [15 points] Suppose c is a constant and the function f is given by:

$$f(x) = \begin{cases} c^2 - x^2, & x < 0 \\ 2(c-x)^2, & x \geq 0 \end{cases}$$

(a) Calculate the following limits:

$$\lim_{x \to 0^-} f(x) \qquad \text{and} \qquad \lim_{x \to 0^+} f(x)$$

Solution:

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} c^2 - x^2 = c^2 - (0)^2 = c^2$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 2(c-x)^2 = 2(c-0)^2 = 2c^2$$

(b) Find a value of the constant c so that the function f is continuous everywhere.

Solution: The function is continuous for $x < 0$ and for $x > 0$. The only place it might be discontinuous is at $x = 0$. For f to be continuous at $x = 0$ we need $\lim_{x \to 0} f(x)$ to exist and to be equal to $f(0)$.

For $\lim_{x \to 0} f(x)$ to exist we need the left hand limit and right hand limit from (a) to be equal. Thus we need

$$c^2 = 2c^2$$

Solving this gives $c = 0$ Hence

$$\lim_{x \to 0} f(x) = 0$$

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Calculating $f(0) = 2(c-0)^2 = 2c^2 = 0$ since $c = 0$. Thus

$$\lim_{x \to 0} f(x) = f(0)$$

and so f is continuous at $x = 0$ if $c = 0$.

3. [10 points] Find the derivative of $f(x) = \sqrt{x+3}$ **using the definition of the derivative**. (No credit will be given for finding the derivative by other means.)

Solution:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h+3}-\sqrt{x+3}}{h}.$$

Multiplying top and bottom by the conjugate $\sqrt{x+h+3}+\sqrt{x+3}$ gives

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\sqrt{x+h+3}-\sqrt{x+3}}{h} \cdot \frac{\sqrt{x+h+3}+\sqrt{x+3}}{\sqrt{x+h+3}+\sqrt{x+3}} \\
&= \lim_{h\to 0} \frac{(x+h+3)-(x+3)}{h(\sqrt{x+h+3}+\sqrt{x+3})} = \lim_{h\to 0} \frac{h}{h(\sqrt{x+h+3}+\sqrt{x+3})} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{x+h+3}+\sqrt{x+3}} = \frac{1}{2\sqrt{x+3}}.
\end{aligned}$$

4. [20 points] Calculate the derivative for each of the following functions:

(a) $(4x^2 + 7x + 3)^{50}$
Solution: Use chain rule, outside function x^{50} and inside function $4x^2 + 7x + 3$.

$$\frac{d((4x^2 + 7x + 3)^{50})}{dx} = 50(4x^2 + 7x + 3)^{49}(8x + 7)$$

(b) $(1 + 2x)^5 \sin(2x^3)$
Solution: First use product rule:
Then,

$$D_x(1 + 2x)^5 = 5(1 + 2x)^4(2) = 10(1 + 2x)^4$$

by chain rule and

$$D_x \sin(2x^3) = \cos(2x^3)(6x^2)$$

by chain rule. So

$$\frac{d((1 + 2x)^5 \sin(2x^3))}{dx} = 10(1+2x)^4(\sin(2x^3))+(1+2x)^5(\cos(2x^3))(6x^2)$$

(c) $3 + \frac{2x}{\sqrt{x+1}}$
Solution:

$$\frac{d(3 + \frac{2x}{\sqrt{x+1}})}{dx} = \frac{2(x + 1)^{1/2} - 2x((1/2)(x + 1)^{-1/2})}{x + 1}$$

(d) $\cos^2(3e^x)$
Solution:

$$\frac{d(\cos^2(3e^x))}{dx} = 2\cos(3e^x)(D_x \cos(3e^x))$$

by chain rule. Then

$$D_x \cos(3e^x) = (-sin(3e^x)(3e^x)$$

by chain rule. So

$$\frac{d(\cos^2(3e^x))}{dx} = 2\cos(3e^x)(-sin(3e^x)(3e^x)$$

5. [10 points] Find the equation of the tangent line to the graph of $y = \tan(2x) + 3\sec x$ at the point $(0, 3)$.

Solution:

$$\frac{dy}{dx} = \sec^2(2x)\frac{d(2x)}{dx} + 3\sec x \tan x = 2\sec^2(2x) + 3\sec x \tan x.$$

Hence at the point $(0, 3)$ where $x = 0$ the slope of the tangent line is

$$\left.\frac{dy}{dx}\right|_{x=0} = 2\sec^2(0) + 3\sec 0 \tan 0 = 2.$$

The tangent line must go through $(0, 3)$ hence it is $y - 3 = 2(x - 0)$ or $y = 2x + 3$.

6. [15 points] Find the maximum and minimum value of $f(x) = \frac{1-x}{x^2+3}$ on $[-2, 1]$. Be sure to show all the steps you need to show in order to justify that your answers really are the maximum and minimum.

Solution: First we need to check that f is continuous on the interval so we can apply our max/min procedure for continuous functions on a closed interval.

The only problem would be when the bottom line becomes zero. But $x^2+3 > 0$ for any x so the function f is continuous on the interval $[-2, 1]$.

Apply the max/min procedure: list x-values at which the max/min might occur.

Endpoints: $x = -2$ and $x = 1$.

Places where derivative is undefined: first calculate the derivative.

$$f'(x) = \frac{-(x^2+3) - (1-x)(2x)}{(x^2+3)^2}$$

Since the denominator, $(x^2+3)^2 > 0$, there are no places where $f'(x)$ is undefined.

Places where derivative is zero: from above we get $-(x^2+3)-(1-x)(2x) = 0$. Rearranging we get:

$$x^2 - 2x - 3 = 0$$

Factorizing gives:

$$(x-3)(x+1) = 0$$

so we have $x = 3$ or $x = -1$. But $x = 3$ is not in our interval.

In total our list of x-values is $x = -2, -1, 1$. The corresponding y-values are $f(-2) = 3/7$, $f(-1) = 1/2$ and $f(1) = 0$.

Thus the maximum value is $1/2$ at $x = -2$ and the minimum is 0 at $x = 1$.

7. [15 points] A rectangle of perimeter 24 inches is rotated about one of its sides to generate a right circular cylinder. What are the dimensions of the rectangle which give a cylinder of maximal volume? (Recall that the volume of a right circular cylinder with height h and radius r is $V = \pi r^2 h$.)

Solution: Let h be the length of the side rotated about in inches and r the length of the other side in inches. Since the perimeter is 24, we have $2h + 2r = 24$ or $h = 12 - r$. Note that h and r must both be nonnegative, hence $0 \leq r \leq 12$. The radius of the resulting cylinder is r and the height is h, hence the volume is

$$V(r) = \pi r^2 h = \pi r^2 (12 - r) = 12\pi r^2 - \pi r^3.$$

We want to maximize this V over the closed interval $0 \leq r \leq 12$. Note that V is a polynomial, hence continuous. We compute

$$\frac{dV}{dr} = 24\pi r - 3\pi r^2$$

which is defined everywhere. Hence the only critical points are where it is zero. Solving $24\pi r - 3\pi r^2 = 0$ gives $r(8-r) = 0$ and $r = 0$ or $r = 8$. (Note that $r = 0$ is already an endpoint.) Thus one of $r = 0$, $r = 8$, or $r = 12$ is the maximum. Since $V(0) = 0 = V(12)$ and $V(8) = \pi 8^2 (12-8) = 256\pi > 0$. The maximum is 256π in^3 for $r = 8$ and $h = 12 - r = 4$.