Math 101 Fall 2000 Exam 2

Instructor: Richard Stong

Tuesday, November 14, 2000

Instructions: This is a closed book, closed notes exam. Use of calculators is not permitted. You have **one hour and fifteen minutes**. Do all 7 problems. Please do all your work on the paper provided.
Please print you name clearly here.

Print name: ____

Upon finishing please sign the pledge below:
On my honor I have neither given nor received any aid on this exam.

____

Grader's use only:

1. ________ /15

2. ________ /10

3. ________ /20

4. ________ /10

5. ________ /10

6. ________ /15

7. ________ /20

1. [15 points] Find the following limits, if they exist.

(a) $\lim_{x\to 3} \frac{x^2-x-6}{x^3-27}$

This limit is indeterminate of type 0/0 so L'Hôpital's Rule gives

$$\lim_{x\to 3} \frac{x^2-x-6}{x^3-27} = \lim_{x\to 3} \frac{2x-1}{3x^2} = \frac{5}{27}.$$

(b) $\lim_{x\to 0} \frac{x-\sin x}{x^3}$

This limit is indeterminate of type 0/0 so L'Hôpital's Rule (applied three times) gives

$$\lim_{x\to 0} \frac{x-\sin x}{x^3} = \lim_{x\to 0} \frac{1-\cos x}{3x^2} = \lim_{x\to 0} \frac{\sin x}{6x} = \lim_{x\to 0} \frac{\cos x}{6} = \frac{1}{6}.$$

(c) $\lim_{x\to 0} (1-x)^{1/(2x)}$

This limit is indeterminate of type 1^∞ so rearranging and using L'Hôpital's Rule gives

$$\begin{aligned}\lim_{x\to 0} (1-x)^{1/(2x)} &= \lim_{x\to 0} \exp\left(\frac{\ln(1-x)}{2x}\right) = \exp\left(\lim_{x\to 0} \frac{\ln(1-x)}{2x}\right) \\ &= \exp\left(\lim_{x\to 0} \frac{\frac{-1}{1-x}}{2}\right) = \exp\left(\frac{-1}{2}\right) = e^{-1/2}.\end{aligned}$$

2. [10 points] Calculate the first three derivatives of the following function

$$g(x) = 3x - \sqrt{x} + \sin(2x)$$

.

$$g' = 3 - \frac{1}{2}x^{-1/2} + 2\cos(2x).$$

$$g''(x) = \frac{1}{4}x^{-3/2} - 4\sin(2x).$$

$$g'''(x) = -\frac{3}{8}x^{-5/2} - 8\cos(2x).$$

3. [20 points] Evaluate the following intergals.

(a) $\int (e^t - 1)^2 dt$

$$\int (e^t - 1)^2 dt = \int (e^{2t} - 2e^t + 1) dt = \frac{1}{2}e^{2t} - 2e^t + t + C.$$

(b) $\int \frac{x^2}{(1-2x^3)^2} dx$

Substituting $u = 1 - 2x^3$ so $du = -6x^2 dx$ gives

$$\int \frac{x^2}{(1-2x^3)^2} dx = -\frac{1}{6} \int u^{-2} du = \frac{1}{6u} + C = \frac{1}{6(1-2x^3)} + C.$$

(c) $\int_1^{\pi^2} \frac{\sin\sqrt{x}}{\sqrt{x}} dx$

Substituting $u = \sqrt{x}$ so $du = \frac{dx}{2\sqrt{x}}$, $x = 1$ means $u = 1$, and $x = \pi^2$ means $u = \pi$ gives

$$\begin{aligned} \int_1^{\pi^2} \frac{\sin\sqrt{x}}{\sqrt{x}} dx = 2 \int_1^{\pi} \sin u \, du &= -2\cos u|_1^{\pi} \\ &= -2\cos\pi + 2\cos 1 = 2 + 2\cos 1. \end{aligned}$$

(d) $\int \frac{(\ln x)^2}{x} dx$

Substituting $u = \ln x$, so $du = \frac{dx}{x}$ gives

$$\int \frac{(\ln x)^2}{x} dx = \int u^2 du = \frac{1}{3}u^3 + C = \frac{1}{3}(\ln x)^3 + C.$$

4. [10 points] Evaluate the definite integral below directly from the definition, that is, by computing $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$ for a regular partition of the given interval of integration.

$$\int_0^2 (2x^2+1)dx$$

For this problem $a = 0$ and $b = 2$ so $\Delta x = (b-a)/n = 2/n$ and $x_i = a + i\Delta x = 2i/n$. Thus

$$\begin{aligned}
\sum_{i=1}^{n} f(x_i)\Delta x &= \sum_{i=1}^{n} \left(2x_i^2+1\right)\Delta x = \sum_{i=1}^{n} \left(2\left(\frac{2i}{n}\right)^2+1\right)\frac{2}{n} \\
&= \sum_{i=1}^{n} \left(\frac{16i^2}{n^3}+\frac{2}{n}\right) = \frac{16}{n^3}\sum_{i=1}^{n} i^2 + \frac{2}{n}\sum_{i=1}^{n} 1 \\
&= \frac{16}{n^3}\left(\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}\right) + 2 = \frac{22}{3}+\frac{8}{n}+\frac{8}{3n^2}.
\end{aligned}$$

Therefore

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \frac{22}{3}+\frac{8}{n}+\frac{8}{3n^2} = \frac{22}{3}.$$

5. [10 points] Find the average value of the function $f(x) = \cos(x/2)$ on the interval $[0, \pi]$.

$$\bar{f} = \frac{1}{\pi - 0} \int_0^{\pi} \cos(x/2) dx = \frac{2}{\pi} \sin(x/2) \Big|_0^{\pi} = \frac{1}{\pi}(2\sin(\pi/2) - 2\sin(0)) = \frac{2}{\pi}.$$

6. [15 points] Find the area of the region in the plane bounded by the curves $y = x + 6$ and $y = x^2$.

The two curves meet when $x+6 = x^2$, that is $0 = x^2 - x - 6 = (x-3)(x+2)$, hence at $x = -2$ and $x = 3$. At $x = 0$, $x + 6 = 6 > x^2 = 0$ so on $[-2, 3]$ the higher curve is the line $y = x + 6$. Hence the area bounded by the curves is

$$A = \int_{-2}^{3} \left((x+6) - x^2\right) dx = \left(\frac{1}{2}x^2 + 6x - \frac{1}{3}x^3\right)\Big|_{-2}^{3}$$
$$= \left(\frac{9}{2} + 18 - 9\right) - \left(2 - 12 + \frac{8}{3}\right) = \frac{125}{6}.$$

7. [20 points] For the function $f(x) = \frac{e^x}{x-1}$, the first two derivatives are $f'(x) = \frac{(x-2)e^x}{(x-1)^2}$ and $f''(x) = \frac{(x^2-4x+5)e^x}{(x-1)^3}$. YOU ARE NOT REQUIRED TO VERIFY THESE FORMULAS. For all other aspects of this problem you are required to justify your answer.

(a) Find all horizontal and vertical asymptotes of the graph $y = e^x/(x-1)$. Be sure to give the limits you need to show these are asymptotes.

$$\lim_{x\to\infty} \frac{e^x}{x-1} = \lim_{x\to\infty} \frac{e^x}{1} = \infty$$
$$\lim_{x\to-\infty} \frac{e^x}{x-1} = \frac{0}{-\infty} = 0.$$

So there is a horizontal asymptote at $y = 0$ that we approach as $x \to -\infty$ only. The only discontinuity is at $x = 1$. Since $e^x > 0$, $f(x) > 0$ for $x > 1$ and $f(x) < 0$ for $x < 1$. Hence

$$\lim_{x\to 1^+} \frac{e^x}{x-1} = \infty, \quad \text{and} \quad \lim_{x\to 1^-} \frac{e^x}{x-1} = -\infty.$$

Therefore we have a vertical asymptote at $x = 1$.

(b) Find the open intervals on which the function f is increasing and those on which it is decreasing.

Since $e^x > 0$ for all x and $(x-1)^2 \geq 0$ for all x, we see f' is negative on $(-\infty, 1)$ and on $(1, 2)$ and $f'(x)$ is positive on $(2, \infty)$. Thus f is decreasing on $(-\infty, 1)$ and on $(1, 2)$ and f is decreasing on $(2, \infty)$.

(c) Find all critical points of $f(x)$ and determine whether they are local maxima, local minima or neither. Justify your answer.

The only place $f'(x)$ is undefined is at $x = 1$, but $f(x)$ is not defined there either so it is not a critical point. The only place where $f'(x) = 0$ is at $x = 2$, so that is the only critical point. By the First Derivative Test, $x = 2$ is a local minimum.

CONTINUED ON NEXT PAGE

(d) Find the intervals on which the function f is concave upward and those on which it is concave downward.

Since $x^2 - 4x + 5 = (x-2)^2 + 1 > 0$ and $e^x > 0$, $f''(x)$ will be negative for $x < 1$ and positive for $x > 1$. Hence f is concave downward on $(-\infty, 1)$ and concave upward on $(1, \infty)$.

(e) Sketch the graph of $y = \frac{e^x}{x-1}$ using your answers to parts (a)-(d) and any additional information required.

SKETCH NOT AVAILABLE