Math 101 Fall 2001 Exam 2 **Solutions**

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Instructions: This is a closed book, closed notes exam. Use of calculators is not permitted. You have **one hour and fifteen minutes**. Do all 8 problems. Please do all your work on the paper provided. You must show your work to receive full credit on a problem. An answer with no supporting work will receive no credit.
Please print you name clearly here.

Print name: ______________________________

Upon finishing please sign the pledge below:
On my honor I have neither given nor received any aid on this exam.

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Grader's use only:

1. ________ /15

2. ________ /20

3. ________ /10

4. ________ /10

5. ________ /15

6. ________ /10

7. ________ /5

8. ________ /15

1. [15 points] Find the following limits, if they exist.

(a) $\lim_{x\to 0} \frac{e^{2x}-1-2x}{x^2}$

This limit is indeterminate of type 0/0 hence applying L'Hopital's rule (twice) gives

$$\lim_{x\to 0} \frac{e^{2x}-1-2x}{x^2} = \lim_{x\to 0} \frac{2e^{2x}-2}{2x} = \lim_{x\to 0} \frac{4e^{2x}}{2} = 2.$$

(b) $\lim_{x\to \pi} \cos(x/2)\cot(x)$

Rewriting this as a ratio using either $\cot x = 1/\tan x$ or $\cot x = \cos x/\sin x$, noting that the result is indeterminate of type ∞/∞ or $0/0$ and applying L'Hopital gives either

$$\lim_{x\to \pi} \cos(x/2)\cot(x) = \lim_{x\to \pi} \frac{\cos(x/2)}{\tan x} = \lim_{x\to \pi} \frac{-(1/2)\sin(x/2)}{\sec^2 x}$$

$$= \frac{-(1/2)}{(-1)^2} = -\frac{1}{2}.$$

or

$$\lim_{x\to \pi} \cos(x/2)\cot(x) = \lim_{x\to \pi} \frac{\cos(x/2)\cos x}{\sin x}$$

$$= \lim_{x\to \pi} \frac{-(1/2)\sin(x/2)\cos x - \cos(x/2)\sin x}{\cos x}$$

$$= \frac{-(1/2)\cdot 1\cdot(-1) - 0\cdot 0}{-1} = -\frac{1}{2}.$$

(c) $\lim_{x\to 0}(1-3x)^{1/(2x)}$

Rewriting the exponential, using continuity of the exponential function to pull the limit into the exponent, and applying L'Hopital's rule gives

$$\lim_{x\to 0}(1-3x)^{1/(2x)} = \lim_{x\to 0} e^{\ln(1-3x)/(2x)} = \exp\left(\lim_{x\to 0} \frac{\ln(1-3x)}{2x}\right)$$

$$= \exp\left(\lim_{x\to 0} \frac{(-3)/(1-3x)}{2}\right) = \exp(-3/2) = e^{-3/2}.$$

2. [20 points] Find the derivatives of the following functions.

(a) $f(x) = x^2 e^{\sin 2x}$

$$\frac{df}{dx} = \frac{d(x^2)}{dx} e^{\sin 2x} + x^2 \frac{d(e^{\sin 2x})}{dx} = 2xe^{\sin 2x} + x^2 e^{\sin 2x} \frac{d(\sin 2x)}{dx}$$
$$= 2xe^{\sin 2x} + 2x^2 \cos(2x) e^{\sin 2x}.$$

(b) $g(t) = \sec(7t^2 + 1)$

$$\frac{dg}{dt} = \sec(7t^2+1)\tan(7t^2+1)\frac{d(7t^2+1)}{dt} = 14t\sec(7t^2+1)\tan(7t^2+1).$$

(c) $F(x) = \arcsin(\ln x)$

$$\frac{dF}{dx} = \frac{1}{\sqrt{1-(\ln x)^2}} \frac{d(lnx)}{dx} = \frac{1}{x\sqrt{1-(\ln x)^2}}.$$

(d) $G(x) = x^{\sqrt{x}}$

Since $G(x) = e^{\sqrt{x}\ln x}$, we have

$$\frac{dG}{dx} = e^{\sqrt{x}\ln x} \frac{d(\sqrt{x}\ln x)}{dx} = e^{\sqrt{x}\ln x}\left(\frac{\ln x}{2\sqrt{x}} + \frac{\sqrt{x}}{x}\right).$$

or since $\ln G(x) = \sqrt{x}\ln x$, using logarithmic differentiation

$$\frac{dG}{dx} = G(x)\frac{d(\sqrt{x}\ln x)}{dx} = x^{\sqrt{x}}\left(\frac{\ln x}{2\sqrt{x}} + \frac{\sqrt{x}}{x}\right).$$

3. [10 points] Find the first three derivatives of the following function:

$$f(x) = \cos(2\sqrt{x})$$

$$f'(x) = -\sin(2\sqrt{x})(x^{-1/2}) = -x^{-1/2}\sin(2\sqrt{x}).$$

$$f''(x) = \frac{1}{2}x^{-3/2}\sin(2\sqrt{x}) - x^{-1}\cos(2\sqrt{x}).$$

$$f'''(x) = -\frac{3}{4}x^{-5/2}\sin(2\sqrt{x}) + \frac{1}{2}x^{-2}\cos(2\sqrt{x}) + x^{-2}\cos(2\sqrt{x}) + x^{-3/2}\sin(2\sqrt{x})$$

$$= \frac{x-3}{4x^{5/2}}\sin(2\sqrt{x}) + \frac{3}{2x^2}\cos(2\sqrt{x}).$$

4. [10 points] The function $y(x)$ is defined (implicitly) by the equation $\sin(x + 2xy) = x^2 + y^2$. Find $\frac{dy}{dx}$.

Differentiating both sides of the equation with respect to x gives

$$\cos(x+2xy)\frac{d(x+2xy)}{dx} = \cos(x+2xy)\left(1 + 2y + 2x\frac{dy}{dx}\right) = 2x + 2y\frac{dy}{dx}.$$

Hence solving for dy/dx gives

$$(2x\cos(x+2xy) - 2y)\frac{dy}{dx} = 2x - (1+2y)\cos(x+2xy)$$

hence

$$\frac{dy}{dx} = \frac{2x - (1+2y)\cos(x+2xy)}{2x\cos(x+2xy) - 2y}.$$

5. [15 points] A kite is flying at an altitude of 80 ft and is carried horizontally by the wind at a rate of 5 ft/sec. At what rate is string released to maintain this flight when 100 ft of string has been released?

Let x be the horizontal distance from the kite's controller to the kite in feet and let s be the length of the string let out in feet. Since the kite is being carried horizontally we know $dx/dt = 5$. Since there is a right triangle in the picture with sides x and 80 and hypotenuse s, we have $s^2 = x^2 + 80^2 = x^2 + 6400$. Differentiating this with respect to time gives

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt}$$

hence

$$\frac{ds}{dt} = \frac{x}{s}\frac{dx}{dt}.$$

When $s = 100$, we have $x^2 = s^2 - 6400 = 10000 - 6400 = 3600$, hence $x = 60$. Therefore

$$\frac{ds}{dt}|_{s=100} = \frac{60}{100} \cdot 5 = 3.$$

Thus string is being let out at a rate of 3 ft/sec.

6. [10 points] A particle moves along the x-axis with acceleration function $a(t) = \sin(t/2)$, initial position $x(0) = 3$, and initial velocity $v(0) = 0$. Find the particle's position $x(t)$ as a function of time.

Since $\frac{d(\cos(t/2))}{dt} = -\frac{1}{2}\sin(t/2)$, we see that $v(t) = C - 2\cos(t/2)$. Since $v(0) = 0 = C - 2\cos(0) = C - 2$, we compute $C = 2$, hence $v(t) = 2 - 2\cos(t/2)$. Since $\frac{d(2t)}{dt} = 2$ and $\frac{d(\sin(t/2))}{dt} = \frac{1}{2}\cos(t/2)$, we see that $x(t) = 2t - 4\sin(t/2) + C$. Since $x(0) = 3 = 2 \cdot 0 - 4\sin(0) + C = C$, we compute $C = 3$. Hence $x(t) = 2t + 3 - 4\sin(t/2)$.

7. [5 points] Express $\sum_{i=1}^{n}(2i-1)^2$ as a polynomial in n.

We compute

$$\sum_{i=1}^{n}(2i-1)^2 = \sum_{i=1}^{n}(4i^2 - 4i + 1) = 4\sum_{i=1}^{n} i^2 - 4\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1$$

$$= 4(\frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n) - 4(\frac{1}{2}n^2 + \frac{1}{2}n) + n = \frac{4}{3}n^3 - \frac{1}{3}n.$$

8. [15 points] For the function $f(x) = \frac{x^2+2x+5}{x+1}$, the first two derivatives are $f'(x) = \frac{(x+3)(x-1)}{(x+1)^2}$ and $f''(x) = \frac{8}{(x+1)^3}$. YOU ARE NOT REQUIRED TO VERIFY THESE FORMULAS. For all other aspects of this problem you are required to justify your answer.

(a) Find the intervals on which the function f is increasing and those on which it is decreasing.

$x+3$ is positive for $x > -3$, $x - 1$ is positive for $x > 1$ and $(x+1)^2$ is positive for $x \neq -1$. Therefore $f'(x)$ is positive on $(-\infty, -3)$ and on $(1, \infty)$ and negative on $(-3, -1)$ and on $(-1, 1)$. Therefore f is increasing on $(-\infty, -3]$, decreasing on $([-3, -1)$, decreasing on $(-1, 1]$, and increasing on $[1, \infty)$.

(b) Find all critical points of $f(x)$ and determine whether they are local maxima, local minima or neither. Justify your answer.

f' is only undefined at $x = -1$ which is not in the domain of f. Thus the only critical points are where $f' = 0$, i.e., at $x = -3$ and $x = 1$. To classify the critical points either:

The derivative switches from positive to negative at $x = -3$, therefore by the First Derivative Test, $x = -3$ is a local maximum. The derivative switches from negative to positive, therefore by the First Derivative Test, $x = 1$ is a local minimum.

or

$f''(-3) = -1$, therefore by the Second Derivative Test, $x = -3$ is a local maximum. $f''(1) = 1$, therefore by the Second Derivative Test, $x = 1$ is a local minimum.

(c) Find the intervals on which the function f is concave upward and those on which it is concave downward.

Since $(x+1)^3$ is negative for $x < -1$ and positive for $x > -1$, we see f'' is negative on $(-\infty, -1)$ and positive on $(-1, \infty)$. Therefore f is concave down on $(-\infty, -1)$ and concave up on $(-1, \infty)$.