Math 101 Fall 2002 Exam 2 **Solutions**

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Instructions: This is a closed book, closed notes exam. Use of calculators is not permitted. You have **one hour and fifteen minutes**. Do all 8 problems. Please do all your work on the paper provided. You must show your work to receive full credit on a problem. An answer with no supporting work will receive no credit.
Please print you name clearly here.

Print name: ______________________________________

Upon finishing please sign the pledge below:
On my honor I have neither given nor received any aid on this exam.

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Grader's use only:

1. ________ /15

2. ________ /10

3. ________ /10

4. ________ /20

5. ________ /5

6. ________ /5

7. ________ /5

8. ________ /20

1. [15 points] Evaluate the following limits, if they exist.

   (a) $\lim_{x \to -3} \frac{x^2+x-6}{x^3-8x+3}$

   This limit is indeterminate of type 0/0, hence L'Hôpital's rule gives

$$\lim_{x \to -3} \frac{x^2+x-6}{x^3-8x+3} = \lim_{x \to -3} \frac{2x+1}{3x^2-8} = \frac{2(-3)+1}{3(-3)^2-8} = -\frac{5}{19}.$$

   (b) $\lim_{t \to 0} \frac{t-\tan t}{t^3}$

   This limit is indeterminate of type 0/0, hence applying L'Hôpital's rule three times gives

$$\begin{aligned}\lim_{t \to 0} \frac{t-\tan t}{t^3} &= \lim_{t \to 0} \frac{1-\sec^2 t}{3t^2} = \lim_{t \to 0} \frac{-2\sec^2 t \tan t}{6t} \\ &= \lim_{t \to 0} \frac{-4\sec^2 t \tan^2 t - 2\sec^4 t}{6} == \frac{-2}{6} = -\frac{1}{3}.\end{aligned}$$

   (c) $\lim_{x \to 0} (1+3x)^{-1/x}$

   This limit is indeterminate of type 1^∞, hence rearranging and then applying L'Hôpital's gives

$$\begin{aligned}\lim_{x \to 0} (1+3x)^{-1/x} &= \lim_{x \to 0} \exp\left(\frac{-\ln(1+3x)}{x}\right) = \exp\left(\lim_{x \to 0} \frac{-\ln(1+3x)}{x}\right) \\ &= \exp\left(\lim_{x \to 0} \frac{-\frac{3}{1+3x}}{1}\right) = \exp(-3) = e^{-3}.\end{aligned}$$

2. [10 points] Find the first three derivatives of the function $f(x) = e^{\cos x}$.

$$f'(x) = -\sin x e^{\cos x}$$

$$f''(x) = -\cos x e^{\cos x} + \sin^2 x e^{\cos x} = (\sin^2 x - \cos x)e^{\cos x}.$$

$$\begin{aligned} f'''(x) &= (2\sin x \cos x + \sin x)e^{\cos x} + (-\sin^3 x + \sin x \cos x)e^{\cos x} \\ &= (-\sin^3 x + 3\sin x \cos x + \sin x)e^{\cos x}. \end{aligned}$$

3. [10 points] A spherical balloon is being inflated at the rate of 32π cm^3/sec. When the radius is 4cm, at what rate is the radius increasing?

Let V be the volume of the ballon in cm^3 and r the radius in cm. Measure time in seconds. We are told $dV/dt = 32\pi$ and we are asked for dr/dt when $r = 4$. The volume and radius are related by

$$V = 4\pi r^3/3$$

hence differentiating implicitly

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

or

$$\frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt}.$$

Plugging in $r = 4$ gives

$$\left.\frac{dr}{dt}\right|_{r=4} = \frac{1}{64\pi}32\pi = \frac{1}{2}.$$

Thus the radius is increasing at 0.5 cm/sec.

4. [20 points] Evaluate the following indefinite integrals:

a. $\int \frac{3}{(2x+1)^3} dx$

$$\int \frac{3}{(2x+1)^3} dx = \frac{-3}{4(2x+1)^2} + C$$

b. $\int 2\cos(3x)(\sin(3x))^4 dx$

Substituting $u = \sin(3x)$, hence $du = 3\cos(3x)dx$ gives

$$\int 2\cos(3x)(\sin(3x))^4 dx = \frac{2}{3}\int u^4 du = \frac{2}{15}u^5 + C = \frac{2}{15}\sin^5(3x) + C.$$

c. $\int \sqrt{x}(1 + x^{3/2})^{1/3} dx$

Substituting $u = 1 + x^{3/2}$, hence $du = (3/2)\sqrt{x}dx$ gives

$$\int \sqrt{x}(1 + x^{3/2})^{1/3} dx = \frac{2}{3}u^{1/3}du = \frac{1}{2}u^{4/3} + C = \frac{1}{2}(1 + x^{3/2})^{4/3} + C.$$

Find the value of the following definite integrals:

d. $\int_1^2 \frac{1+(1/t)}{t^2} dt$

$$\int_1^2 \frac{1 + (1/t)}{t^2} dt = \int_1^2 (t^{-2} + t^{-3})dt = \left(-\frac{1}{t} - \frac{1}{2t^2}\right)\Bigg|_1^2$$

$$= \left(-\frac{1}{2} - \frac{1}{8}\right) - \left(-\frac{1}{1} - \frac{1}{2}\right) = \frac{7}{8}.$$

e. $\int_0^{\ln 3} \frac{e^x}{1+2e^x} dx$

Substituting $u = 1 + 2e^x$, hence $du = 2e^x dx$, at $x = 0$, $u = 3$, and at $x = \ln 3$, $u = 7$, gives

$$\int_0^{\ln 3} \frac{e^x}{1 + 2e^x} dx = \frac{1}{2}\int_3^7 \frac{1}{u} du = \frac{1}{2}\ln u\Big|_3^7 = \frac{1}{2}(\ln 7 - \ln 3).$$

5. [5 points] Solve $\frac{dy}{dx} = (x-1)^9, y(0) = 0$

An antiderivative of $(x-1)^9$ is $\frac{1}{10}(x-1)^{10}$, hence $y(x) = \frac{1}{10}(x-1)^{10} + C$ for some constant C. Plugging in $x = 0$ gives $0 = y(0) = \frac{1}{10}(-1)^{10} + C = \frac{1}{10} + C$. Thus $C = -\frac{1}{10}$ and $y(x) = \frac{1}{10}(x-1)^{10} - \frac{1}{10}$.

6. [5 points] Differentiate the following function $f(x) = \int_2^{\sin(x)} (\ln t)^2 dt$

Let $g(u) = \int_2^u (\ln t)^2 dt$, then $\frac{dg}{du} = (\ln u)^2$ and $f(x) = g(\sin x)$. Thus

$$\frac{df}{dx} = g'(\sin x)\frac{d\sin x}{dx} = (\ln \sin x)^2 \cos x.$$

7. [5 points] Express $\sum_{i=1}^n (i+2)(i-1)^2$ as a polynomial in n.

Expanding the summand gives $(i+2)(i-1)^2 = i^3 - 3i + 2$, hence

$$\begin{aligned}\sum_{i=1}^n (i+2)(i-1)^2 = \sum_{i=1}^n (i^3 - 3i + 2) &= \sum_{i=1}^n i^3 - 3\sum_{i=1}^n i + 2\sum_{i=1}^n 1 \\ &= \left(\frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2\right) - 3\left(\frac{1}{2}n^2 + \frac{1}{2}n\right) + 2n \\ &= \frac{1}{4}n^4 + \frac{1}{2}n^3 - \frac{5}{4}n^2 + \frac{1}{2}n.\end{aligned}$$

8. [20 points] For the function $f(x) = (x^2+7)e^{-x/4}$, the first two derivatives are $f'(x) = -\frac{1}{4}(x-1)(x-7)e^{-x/4}$ and $f''(x) = \frac{1}{16}(x-3)(x-13)e^{-x/4}$.

(a) Find (and justify) all horizontal and vertical asymptotes of the graph $y = f(x)$. At the vertical asymptotes compute both the left and right hand limits of $f(x)$.

This function is continuous everywhere, hence there are no vertical asymptotes. For horizontal asymptotes we compute

$$\lim_{x\to\infty} \frac{x^2+7}{e^{x/4}} = \lim_{x\to\infty} \frac{2x}{(1/4)e^{x/4}} = \lim_{x\to\infty} \frac{2}{(1/16)e^{x/4}} = 0$$

and

$$\lim_{x\to-\infty} \frac{x^2+7}{e^{x/4}} = +\infty$$

Thus there is a horizontal asymptote $y = 0$ which the graph approaches as x tends to $+\infty$ only.

(b) Find the intervals on which $f(x)$ is increasing and those on which it is decreasing.

Since $x - 1$ is positive exactly for $x > 1$, $x - 7$ is positive exactly for $x > 7$, and $e^{-x/4}$ is always positive, we see $f'(x)$ is negative on $(-\infty, 1)$ and on $(7, \infty)$ and positive on $(1, 7)$. Thus f is increasing on $[1, 7]$ and is decreasing on $(-\infty, 1]$ and on $[7, \infty)$.

(c) Find the critical points of $f(x)$ and classify them as local maxima, local minima or neither.

The derivative is always defined and is only zero at $x = 1$ and at $x = 7$, thus these are the critical points. At $x = 1$, the function switches from decreasing to increasing, so $x = 1$ is a local minimum. At $x = 7$, the function switches from increasing to decreasing, so $x = 7$ is a local maximum.

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(d) Find the intervals on which $f(x)$ is concave upward and those on which it is concave downward.

Since $x - 3$ is positive exactly for $x > 3$, $x - 13$ is positive exactly for $x > 13$, and $e^{-x/4}$ is always positive, we see $f''(x)$ is positive on $(-\infty, 3)$ and on $(13, \infty)$ and negative on $(3, 13)$. Thus f is concave upward on $(-\infty, 3)$ and on $(13, \infty)$ and concave downward on $(3, 13)$.

(e) Sketch the graph of $y = (x^2 + 7)e^{-x/4}$ showing the results of (a)-(d). (The following values may be helpful $f(1) \approx 6.23$, $f(3) \approx 7.56$, $f(7) \approx 9.73$, $f(13) \approx 6.82$.)