Midterm I Exam, Spring 2005

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Instructions: This is a closed book, closed notes exam. Use of calculators is not permitted. You have **one hour**. Do all 7 problems. Please do all your work on the paper provided. You must show your work to receive full credit on a problem. **An answer with no supporting work will receive no credit**.

Please print your name clearly here:

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Upon finishing please sign the pledge below:
On my honor, I have neither given nor received any aid on this exam.

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Grader's use only:

1. ..../15
2. ..../10
3. ..../10
4. ..../20
5. ..../15
6. ..../15
7. ..../15

1. *(15 points)* Evaluate each of the following limits:

(a) $\lim_{x \to -5} \frac{5+x}{25-x^2}$

$$\begin{aligned} &= \lim_{x \to -5} \frac{5+x}{(5+x)(5-x)} \\ &= \lim_{x \to -5} \frac{1}{5-x} \\ &= \frac{1}{10} \end{aligned}$$

(b) $\lim_{t \to 0} \frac{\sin 5t + \sin t \cos t}{t}$

$$\begin{aligned} &= \lim_{t \to 0} \frac{\sin 5t}{t} + \lim_{t \to 0} \frac{\sin t \cos t}{t} \\ &= \lim_{t \to 0} 5\frac{\sin 5t}{5t} + \lim_{t \to 0} \frac{\sin t}{t} \cos t \\ &= 5+1 \\ &= 6 \end{aligned}$$

(c) $\lim_{x \to 0} \frac{2-\sqrt{x+4}}{x}$

$$\begin{aligned} &= \lim_{x \to 0} \frac{(2-\sqrt{x+4})(2+\sqrt{x+4})}{x(2+\sqrt{x+4})} \\ &= \lim_{x \to 0} \frac{4-(x+4)}{x(2+\sqrt{x+4})} \\ &= \lim_{x \to 0} \frac{-1}{(2+\sqrt{x+4})} \\ &= \frac{-1}{4} \end{aligned}$$

2. *(10 points)*Find the derivative of the function, $f(x) = 3x^2 + 4$, using the definition of the derivative. No credit will be given for computing the derivative in any other way.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{3(x+h)^2 + 4 - (3x^2+4)}{h} \\ &= \lim_{h\to 0} \frac{3x^2 + 6xh + 3h^2 + 4 - (3x^2+4)}{h} \\ &= \lim_{h\to 0} \frac{6xh + 3h^2}{h} \\ &= \lim_{h\to 0} 6x + h \\ &= 6x \end{aligned}$$

3. *(10 points)* Show that $5x^3 - 7x + 12 = 0$ has a solution on $[-2, 1]$. Clearly explain your reasoning.

$$\begin{aligned} f(x) &= 5x^3 - 7x + 12 \\ \\ f(-2) &= 5(-2)^3 - 7(-2) + 12 \\ &= -40 + 26 \\ &= -14 \\ \\ f(1) &= 5(1)^3 - 7(1) + 12 \\ &= 10 \end{aligned}$$

Since $-14 \leq 0 \leq 10$, by the Intermediate Value Theorem, $f(x)$ has a solution on $[-2, 1]$.

4. *(20 points)* Evaluate the derivatives of the following functions:

(a) $f(x) = (\tan x)(x^3 - 7)^2$

$$\begin{aligned} f'(x) &= D_x(\tan x)(x^3-7)^2 + (\tan x)D_x[(x^3-7)^2] \\ &= (\sec^2 x)(x^3-7)^2 + (\tan x)2(x^3-7)(3x^2) \\ &= (\sec^2 x)(x^3-7)^2 + 6x^2(x^3-7)(\tan x) \end{aligned}$$

(b) $f(x) = \ln\left[\dfrac{(7x^2+1)^3}{(x-7)^2}\right]$

$$\begin{aligned} f(x) &= \ln(7x^2+1)^3 - \ln(x-7)^2 \\ &= 3\ln(7x^2+1) - 2\ln(x-7) \end{aligned}$$

$$\begin{aligned} f'(x) &= D_x[3\ln(7x^2+1)] - D_x[2\ln(x-7)] \\ &= 3\frac{D_x(7x^2+1)}{(7x^2+1)} - 2\frac{D_x(x-7)}{(x-7)} \\ &= 3\frac{14x}{7x^2+1} - 2\frac{1}{x-7} \\ &= \frac{52x}{7x^2+1} - \frac{2}{x-7} \end{aligned}$$

(c) $x(t) = \dfrac{\sqrt{t-3}}{\sin^2 t}$

$$\begin{aligned} x'(t) &= \frac{D_t(\sqrt{t-3})(\sin^2 t) - D_t(\sin^2 t)(\sqrt{t-3})}{(\sin^2 t)^2} \\ &= \frac{\frac{1}{2}(t-3)^{\frac{-1}{2}}(\sin^2 t) - 2\sin t\cos t(\sqrt{t-3})}{\sin^4 t} \\ &= \frac{\frac{\sin t}{2\sqrt{t-3}} - 2\cos t\sqrt{t-3}}{\sin^3 t} \\ &= \frac{\sin t - 4\cos t(t-3)}{2\sqrt{t-3}\sin^3 t} \end{aligned}$$

5. *(15 points)* Find the equation of the line tangent to $f(x) = (x^3 + 4x - 1)e^{x^2}$ at the point $(0, -1)$.

$$\begin{aligned} f'(x) &= D_x(x^3 + 4x - 1)(e^{x^2}) + D_x(e^{x^2})(x^3 + 4x - 1) \\ &= (3x^2 + 4)(e^{x^2}) + (e^{x^2})(2x)(x^3 + 4x - 1) \end{aligned}$$

$$\begin{aligned} f'(0) &= (3(0)^2 + 4)(e^{(0)^2}) + (e^{(0)^2})(2(0))((0)^3 + 4(0) - 1) \\ &= 4 \end{aligned}$$

$$m = 4$$

$$\begin{aligned} y - (-1) &= 4(x - 0) \\ y &= 4x + 1 \end{aligned}$$

6. *(15 points)* Given the function:

$$f(x) = \begin{cases} 3x^3 - \frac{4}{x} & x < 2 \\ 20 & x = 2 \\ 7x^2 - 4x + 2 & x > 2 \end{cases}$$

(a) Find the following limits: $\lim_{x \to 2^-} f(x)$ and $\lim_{x \to 2^+} f(x)$

$$\begin{aligned} \lim_{x \to 2^-} f(x) &= \lim_{x \to 2^-} 3x^3 - \frac{4}{x} \\ &= 3(2)^3 - \frac{4}{2} \\ &= 22 \end{aligned}$$

$$\begin{aligned} \lim_{x \to 2^+} f(x) &= \lim_{x \to 2^+} 7x^2 - 4x + 2 \\ &= 7(2)^2 - 4(2) + 2 \\ &= 22 \end{aligned}$$

(b) Is $f(x)$ continuous at $x = 2$? Why or why not?

Since

$$\begin{aligned} \lim_{x \to 2^-} f(x) &= 22 \\ \text{and } \lim_{x \to 2^+} f(x) &= 22 \end{aligned}$$

then

$$\lim_{x \to 2} f(x) = 22$$

However,

$$f(2) = 20 \neq 22$$

Therefore, $f(x)$ is not continuous at $x = 2$.

7. *(20 points)* A rectangular box with bottom, but no top is to be made out of 54 square inches of material. The length of the box is to be twice the width. If w, l, and h are the width, length and height, respectively, of the box, find the maximum possible volume of the box and the dimensions of the box. (Here, $V = wlh$, and note that $\sqrt{27} \approx 5.196$)

$$\begin{aligned} V &= wlh \\ &= 2w^2h \end{aligned}$$

$$\begin{aligned} SA &= 2wh + 2hl + wl \\ &= 2wh + 2h(2w) + w(2w) \\ 54 &= 2wh + 4wh + 2w^2 \\ &= 6wh + 2w^2 \\ \text{So, if } h = 0, \text{ then } w = \sqrt{27} & \\ \text{Thus, } w \epsilon [0, \sqrt{27}] & \\ V &= w(2w)\left(\frac{54 - 2w^2}{6w}\right) \\ &= 18w - \frac{2w^3}{3} \end{aligned}$$

$$\begin{aligned} V'(w) &= 18 - 2w^2 \\ v'(w) &= 0 \\ w &= 3 \end{aligned}$$

Testing endpoints and critical points, max dimension:

$$\begin{aligned} w &= 3 \\ l &= 6 \\ h &= 2 \\ V &= 36 \end{aligned}$$