Midterm II Exam, Spring 2005

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Instructions: This is a closed book, closed notes exam. Use of calculators is not permitted. You have **one hour**. Do all 7 problems. Please do all your work on the paper provided. You must show your work to receive full credit on a problem. **An answer with no supporting work will receive no credit**.

Please print your name clearly here:

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Upon finishing please sign the pledge below:
On my honor, I have neither given nor received any aid on this exam.

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Grader's use only:

1. ..../10
2. ..../10
3. ..../10
4. ..../10
5. ..../5
6. ..../10
7. ..../20

1. *(10 points)*Find the equation of the line tangent to the curve

$$x^2y^2 - 2xy + x = 1$$

at the point $(1, 2)$

$$\begin{aligned}
D_x(x^2y^2 - 2xy + x &= 1) \\
2xy^2 + x^2(2yy') - 2y - 2xy' + 1 &= 0 \\
(2x^2y - 2x)y' &= 2y - 2xy^2 - 1 \\
y' &= \frac{2y - 2xy^2 - 1}{2x^2y - 2x} \\
y'(1,2) &= \frac{2(2) - 2(1)(2)^2 - 1}{2(1)^2(2) - 2(1)} \\
y' &= \frac{-5}{2}
\end{aligned}$$

So, the equation of the line is:

$$y - 2 = \frac{-5}{2}(x - 1)$$

2. *(10 points)* Evaluate the following limits:

(a) $\lim_{x \to 0} \dfrac{\cos x + x - 1}{3x}$

$$\begin{aligned} &= \frac{0}{0} \\ &= \lim_{x\to 0} \frac{-\sin x + 1}{3} \\ &= \frac{1}{3} \end{aligned}$$

(b) $\lim_{x \to 0^+} (e^x - 1)^x$

$$\begin{aligned} &= 0^0 \\ y &= \lim_{x\to 0^+} (e^x - 1)^x \\ \ln y &= \lim_{x\to 0^+} \ln((e^x - 1)^x) \\ &= \lim_{x\to 0^+} x \ln(e^x - 1) \\ &= 0 * -\infty \\ &= \lim_{x\to 0^+} \frac{\ln(e^x - 1)}{\frac{1}{x}} \\ &= \lim_{x\to 0^+} \frac{-x^2 e^x}{e^x - 1} \\ &= \frac{0}{0} \\ &= \lim_{x\to 0^+} \frac{-2xe^x - x^2 e^x}{e^x} \\ &= 0 \\ y &= e^0 \\ &= 1 \end{aligned}$$

3. *(10 points)* The edges of a cube (all sides are equal length) are increasing at the rate of 2 in/min. At what rate is the volume of the cube increasing when the edges are 4 inches?

$$\frac{dx}{dt} = 2 \text{ in/min}$$
$$V = x^3$$

$$\frac{dV}{dt} = 3x^2 \frac{dx}{dt}$$

At $x = 4$,

$$\frac{dV}{dt} = 3(4)^2(2)$$
$$\frac{dV}{dt} = 96 \text{ in}^3\text{/min}$$

4. *(15 points)* Evaluate the following integrals:

(a) $\int \frac{x}{1+x^2}dx$

$$\begin{aligned}
\text{Let } u &= 1+x^2\\
du &= 2xdx\\
\frac{1}{2}du &= xdx\\
\int \frac{x}{1+x^2}dx &= \int \frac{du}{2u}\\
&= \frac{1}{2}\ln(|u|)+C\\
&= \frac{1}{2}\ln(1+x^2)+C
\end{aligned}$$

(b) $\int_0^{\frac{\pi}{6}} \frac{\sin 3x}{\cos^5 3x}dx$

$$\begin{aligned}
\text{Let } u &= \cos 3x\\
du &= -3\sin 3x dx\\
\text{if } x &= 0,\ \text{then } u=1\\
\text{if } x &= \frac{\pi}{6},\ \text{then } u=0\\
\int_0^{\frac{\pi}{6}} \frac{\sin 3}{\cos^5 3x}dx &= \int_1^0 \frac{-du}{3u^5}\\
&= \left[-\frac{1}{3*(-4)}u^{-4}\right]_1^0\\
&= \frac{1}{12}((\frac{1}{0})-1)\\
&= \infty
\end{aligned}$$

Note: I gave credit for answers of $\frac{-1}{12}$ since it was not my intention to give a definite integral with infinity as the answer.

5. *(5 points)* Use the Fundamental Theorem of Calculus to determine the derivative of $f(x) = \int_{12}^{x} t^5(e^t + 4t)^3 dt$.

By the Fundamental Theorem of Calculus, if

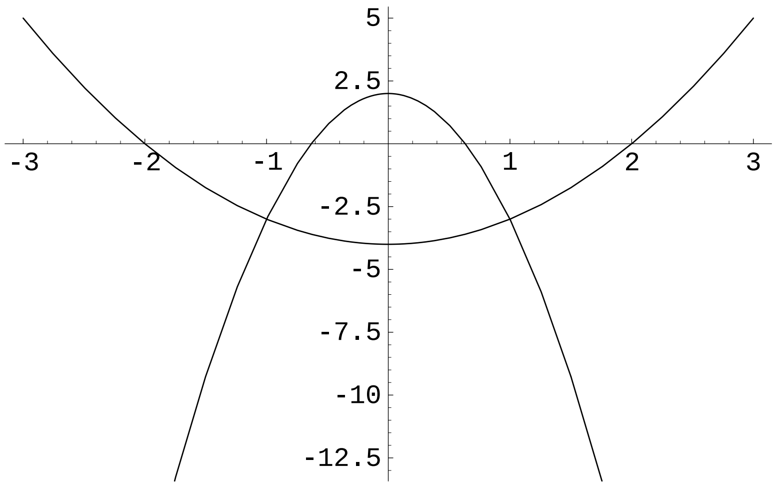
$$f(x) = \int_{a}^{x} g(t)dt$$

then

$$f'(x) = g(x)$$

So, in this problem, $f'(x) = x^5(e^x + 4x)^3$

6. *(10 points)* Find the area between the functions $f(x) = x^2 - 4$ and $g(x) = 2 - 5x^2$.



Note: the top function is $g(x)$ and the bottom function is $f(x)$. To find the intersection points:

$$\begin{aligned} x^2 - 4 &= 2 - 5x^2 \\ 6x^2 &= 6 \\ x^2 &= 1 \\ x &= -1, 1 \end{aligned}$$

So, the area between $f(x)$ and $g(x)$ is:

$$\begin{aligned} \int_{-1}^{1} [g(x) - f(x)]dx &= \int_{-1}^{1} (2 - 5x^2 - x^2 + 4)dx \\ &= \int_{-1}^{1} (6 - 6x^2)dx \\ &= \left[6x - 2x^3\right]_{-1}^{1} \\ &= (6 - 2) - (-6 + 2) \\ &= 8 \end{aligned}$$

7. *(20 points)*For the function $f(x)$ with given first and second derivatives:

$$\begin{aligned} f(x) &= \frac{x^2+x-2}{x^2} \\ f'(x) &= \frac{4-x}{x^3} \\ f''(x) &= \frac{2(x-6)}{x^4} \end{aligned}$$

Be sure to clearly answer each of the items below for full credit.

(a) Find the intervals on which $f(x)$ is increasing and those on which it is decreasing.

Given $f'(x) = \dfrac{4-x}{x^3}$, the critical points are $x = 0, x = 4$

Interval	$f'(x)$	$f(x)$
$(-\infty, 0)$	< 0	decreasing
$(0, 4)$	> 0	increasing
$(4, \infty)$	< 0	decreasing

(b) Determine critical points for $f(x)$, and classify them as local maxima, local minima or neither. Determine the points of inflection for $f(x)$, if any.

From (a), the candidates are $x = 0, x = 4$. Note that $f(x)$ is undefined at $x = 0$, so it is neither. From the first derivative test, $f(x)$ has a local max at $x = 4$.

Using the second derivative confirms that $x = 6$ is the only point of inflection.

(c) Determine the horizontal and vertical asymptotes of $f(x)$.

$f(x)$ is undefined at $x = 0$, so the line $x = 0$ is a vertical asymptote.

$$\begin{aligned} \lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{x^2+x-2}{x^2} \\ &= \lim_{x\to\infty} 1 + \frac{1}{x} - \frac{2}{x^2} \\ &= 1 \end{aligned}$$

Similarly,

$$\lim_{x\to-\infty} f(x) = 1$$

So, $f(x)$ has a horizontal asymptote at $y = 1$.

(d) Sketch the graph of $f(x)$ showing the results of (a) - (c).