Math 101 Fall 2001 Final Exam **Solutions**

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Instructions: This is a closed book, closed notes exam. Use of calculators is not permitted. You have **three hours**. Do all 12 problems. Please do all your work on the paper provided.
Please print you name clearly here.

Print name: ______________________________________

Upon finishing please sign the pledge below:
On my honor I have neither given nor received any aid on this exam.

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Grader's use only:

1. ________ /15

2. ________ /30

3. ________ /20

4. ________ /30

5. ________ /20

6. ________ /10

7. ________ /10

8. ________ /15

9. ________ /15

10. ________ /10

11. ________ /10

12. ________ /15

1. [15 points] Evaluate the following limits, if they exist.

   (a) $\lim\limits_{x\to 3} \frac{x^2-4x+3}{x^2-x-6}$

   The limit is indeterminate of type 0/0, therefore applying L'Hopital's rule gives

$$\lim_{x\to 3} \frac{x^2-4x+3}{x^2-x-6} = \lim_{x\to 3} \frac{2x-4}{2x-1} = \frac{2}{5}.$$

   or

$$\lim_{x\to 3} \frac{x^2-4x+3}{x^2-x-6} = \lim_{x\to 3} \frac{(x-3)(x-1)}{(x-3)(x+2)} = \lim_{x\to 3} \frac{x-1}{x+2} = \frac{2}{5}.$$

   (b) $\lim\limits_{x\to 0} \frac{e^{x^2}-1}{1-\cos x}$

   This limit is indeterminate of type 0/0, hence applying L'Hopital's rule (twice) gives

$$\lim_{x\to 0} \frac{e^{x^2}-1}{1-\cos x} = \lim_{x\to 0} \frac{2xe^{x^2}}{\sin x} = \lim_{x\to 0} \frac{2e^{x^2}+4x^2e^{x^2}}{\cos x} = \frac{2}{1} = 2.$$

   (c) $\lim\limits_{x\to 1} x^{2/(x-1)}$

   This limit is indeterminate of type 1^∞, therefore rewriting and applying L'Hopital's rule gives:

$$\lim_{x\to 1} x^{2/(x-1)} = \lim_{x\to 1} \exp\left(\frac{2\ln x}{x-1}\right) = \exp\left(\lim_{x\to 1} \frac{2\ln x}{x-1}\right)$$

$$= \exp\left(\lim_{x\to 1} \frac{2/x}{1}\right) = \exp(2) = e^2.$$

2. [30 points] Compute the derivatives of the following functions.

(a) $f(x) = x^3 \ln(x+3)$

$$\frac{df}{dx} = 3x^2 \ln(x+3) + \frac{x^3}{x+3}.$$

(b) $g(t) = \sec(e^t)$

$$\frac{dg}{dt} = \sec(e^t)\tan(e^t)\frac{d(e^t)}{dt} = e^t \sec(e^t)\tan(e^t).$$

(c) $h(w) = \arctan(3w^2)$

$$\frac{dh}{dw} = \frac{1}{1+(3w^2)^2}\frac{d(3w^2)}{dw} = \frac{6w}{1+9w^4}.$$

(d) $G(x) = \int_1^{x^2} \frac{\sin t}{t} dt$

$$\frac{dG}{dx} = \frac{\sin(x^2)}{x^2}\frac{d(x^2)}{dx} = \frac{2\sin(x^2)}{x}.$$

(e) $H(z) = (2 + (1+3\sin z)^3)^{1/2}$

$$\begin{aligned}\frac{dH}{dz} &= \frac{1}{2}(2+(1+3\sin z)^3)^{-1/2}\frac{d(2+(1+3\sin z)^3)}{dz} \\ &= \frac{3}{2}(2+(1+3\sin z)^3)^{-1/2}(1+3\sin z)^2\frac{d(3\sin z)}{dz} \\ &= \frac{9}{2}(2+(1+3\sin z)^3)^{-1/2}(1+3\sin z)^2\cos z.\end{aligned}$$

3. [20 points] You want to build a rectangular box with no top, a square base and a volume of 500 cm^3. What dimensions will minimize the total surface area? Be sure to justify that your answer is really a global minimimum.

   Let x be the length of a side of the base of the box in cm and h the height in cm. The volume is $V = x^2h = 500$, hence $h = 500/x^2$. Since x and h are lengths, they must be positive so $0 < x < \infty$. The surface area is $S(x) = x^2 + 4xh = x^2 + 2000/x$. Then $S'(x) = 2x - 2000/x^2 = 2(x^3 - 1000)/x^2$. The only critical point is when $x = 10$. Since $S'(x) < 0$ for $x < 10$ and $S'(x) > 0$ for $x > 10$ or because $S''(x) = 2 + 4000/x^3 > 0$ for all x, $x = 10$ is a global minimum. Therefore the dimensions of the minimal surface area box are $x = 10$ cm and $h = 5$ cm.

4. [30 points] Let $f(x) = \frac{x^2-3}{x^2-1}$.

(a) Find all horizontal and vertical asymptotes of the graph $y = f(x)$. At the vertical asymptotes compute both the left and right hand limits of $f(x)$.

$$\lim_{x\to\infty} \frac{x^2-3}{x^2-1} = 1 \quad \text{and} \quad \lim_{\mathrm{x}\to-\infty} \frac{\mathrm{x}^2-3}{\mathrm{x}^2-1} = 1$$

so $y = 1$ is the only horizontal asymptote. The vertical asymptotes are at $x = \pm 1$ where we get constant/0. Since $f > 0$ for $-1 < x < 1$ and $f > 0$ for $1 < x < \sqrt{3}$ and for $-\sqrt{3} < x < -1$ we have

$$\lim_{x\to-1^-} \frac{x^2-3}{x^2-1} = -\infty, \qquad \lim_{x\to-1^+} \frac{x^2-3}{x^2-1} = \infty,$$

$$\lim_{x\to1^-} \frac{x^2-3}{x^2-1} = \infty, \qquad \lim_{x\to1^+} \frac{x^2-3}{x^2-1} = -\infty$$

(b) Find the intervals on which $f(x)$ is increasing and those on which it is decreasing.

$$\frac{df}{dx} = \frac{2x(x^2-1) - 2x(x^2-3)}{(x^2-1)^2} = \frac{4x}{(x^2-1)^2}$$

Note f is discontinuous at $x = \pm 1$. The denominator is always nonnegative, the numerator is positive for $x > 0$ and negative for $x < 0$. Hence f is decreasing on $(-\infty, -1)$ and on $(-1, 0]$ and f is increasing on $[0, 1)$ and on $(1, \infty)$.

(c) Find the critical points of $f(x)$ and determine if they are local maxima or local minima.

Since $f'(x)$ exists for $x \neq \pm 1$ and $f'(x) = 0$ only when the numerator is zero, i.e., when $x = 0$, the only critical point of f is at $x = 0$. Since f switches from decreasing to increasing at $x = 0$, $x = 0$ is a local minimum by the First Derivative Test or from part (d) below $f''(0) = 4 > 0$ so by the Second Derivative Test, $x = 0$ is a local minimum.

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(d) Find the intervals on which $f(x)$ is concave upward and those on which it is concave downward.

$$f''(x) = \frac{4(x^2-1)^2 - 4x \cdot 4x(x^2-1)}{(x^2-1)^4} = \frac{-4(3x^2+1)}{(x^2-1)^3}.$$

Since $3x^2+1>0$, $f''(x)<0$ for $|x|>1$ and $f''(x)>0$ for $-1<x<1$, f is concave down on $(-\infty,-1)$ and on $(1,\infty)$ and f is concave up on $(-1,1)$.

(e) Sketch the graph of $y = \frac{x^2-3}{x^2-1}$ using your results in parts (a)-(d).

5. [20 points] Evaluate the following integrals.

(a) $\int (x^2+3)^2 dx$

$$\int (x^2+3)^2 dx = \int (x^4+6x^2+9)dx = \frac{1}{5}x^5 + 2x^3 + 9x + C.$$

(b) $\int_0^{\pi/2} \frac{\cos x}{2+\sin x} dx$

Substituting $u = 2+\sin x$, $du = \cos x dx$, when $x = 0$, $u = 2$ and when $x = \pi/2$, $u = 3$

$$\int_0^{\pi/2} \frac{\cos x}{2+\sin x} dx = \int_2^3 \frac{1}{u} du = \ln u|_2^3 = \ln 3 - \ln 2.$$

(c) $\int \sec^2 x e^{\tan x} dx$

Substituting $u = \tan x$, $du = \sec^2 x dx$ so

$$\int \sec^2 x e^{\tan x} dx = \int e^u du = e^u + C = e^{\tan x} + C.$$

(d) $\int \frac{x^3}{\sqrt{1-4x^8}} dx$

Substituting $u = 2x^4$, $du = 8x^3 dx$ gives

$$\int \frac{x^3}{\sqrt{1-4x^8}} dx = \frac{1}{8} \int \frac{1}{\sqrt{1-u^2}} du = \frac{1}{8} \arcsin u + C = \frac{1}{8} \arcsin(2x^4) + C.$$

6. [10 points] Evaluate $\int_0^2 (1+3x^2)dx$ by computing $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$. No credit will be given for computing the integral in any other manner.

We have $a = 0$, $b = 2$, hence $\Delta x = (b-a)/n = 2/n$ and $x_i = a + i\Delta x = 2i/n$. Thus

$$\sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} f(\frac{2i}{n})\frac{2}{n} = \sum_{i=1}^{n} \left(1 + \frac{12i^2}{n^2}\right)\frac{2}{n} = \frac{2}{n}\sum_{i=1}^{n} 1 + \frac{24}{n^3}\sum_{i=1}^{n} i^2$$

$$= 2 + \frac{24}{n^3}\left(\frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n\right) = 10 + \frac{12}{n} + \frac{4}{n^2}.$$

Therefore

$$\int_0^2 (1+3x^2)dx = \lim_{n\to\infty}\left(10 + \frac{12}{n} + \frac{4}{n^2}\right) = 10.$$

7. [10 points] Find the area of the region in the plane bounded by $y = 4 - x^2$ and $y = x^2 - 2x - 8$.

The two parabolas cross where $4 - x^2 = x^2 - 2x - 8$, hence $2x^2 - 2x - 12 = 2(x-3)(x+2) = 0$ or $x = -2$ and $x = 3$. For $-2 < x < 3$, $4 - x^2 > x^2 - 2x - 8$. Hence $a = -2$, $b = 3$, $f(x) = 4 - x^2$, $g(x) = x^2 - 2x - 8$ and the area is

$$A = \int_{-2}^{3} ((4 - x^2) - (x^2 - 2x - 8))dx = \int_{-2}^{3} (12 + 2x - 2x^2)dx$$

$$= \left(12x + x^2 - \frac{2}{3}x^3\right)\Big|_{-2}^{3} = (36 + 9 - 18) - \left(-24 + 4 + \frac{16}{3}\right) = \frac{125}{3}.$$

8. [15 points] Suppose a particle on a line has velocity function $v(t) = t^2 - 4t + 3$ for $1 \le t \le 4$. Find the net distance travelled by the particle between $t = 1$ and $t = 4$ and the total distance travelled between $t = 1$ and $t = 4$.

$$\text{Net} = \int_1^4 v(t)dt = \int_1^4 (t^2 - 4t + 3)dt = \left(\frac{1}{3}t^3 - 2t^2 + 3t\right)\Big|_1^4$$
$$= \left(\frac{64}{3} - 32 + 12\right) - \left(\frac{1}{3} - 2 + 3\right) = \frac{4}{3} - \frac{4}{3} = 0.$$

Since $t^2 - 4t + 3 = (t-1)(t-3) \le 0$ for $1 \le t \le 3$ and $t^2 - 4t + 3 = (t-1)(t-3) \ge 0$ for $3 \le t \le 4$

$$\text{Total} = \int_1^4 |v(t)|dt = \int_1^3 (-t^2 + 4t - 3)dt + \int_3^4 (t^2 - 4t + 3)dt$$
$$= \left(-\frac{1}{3}t^3 + 2t^2 - 3t\right)\Big|_1^3 + \left(\frac{1}{3}t^3 - 2t^2 + 3t\right)\Big|_3^4 =$$
$$\left\{(-9 + 18 - 9) - \left(-\frac{1}{3} + 2 - 3\right)\right\} + \left\{\left(\frac{64}{3} - 32 + 12\right) - (9 - 18 + 9)\right\}$$
$$= \frac{4}{3} + \frac{4}{3} = \frac{8}{3}.$$

9. [15 points] Let R be the region in the plane bounded by $x = 4y - y^2$ and the y-axis. Let S be the solid of revolution that results from revolving R about the y-axis. Express the volume of S as a definite integral in TWO ways, using the method of washers and the method of shells. Evaluate ONE of the two integrals (your choice).

The y-axis is defined by $x = 0$ so the parabola crosses the y-axis at $(0, 0)$ and $(0, 4)$. For $0 \leq y \leq 4$, $4y - y^2 \geq 0$. Thus R is the region to the left of $x = 4y - y^2$, to the right of $x = 0$ between $y = 0$ and $y = 4$. Thus washers gives

$$V = \int_0^4 \pi \left((4y - y^2)^2 - 0\right) dy = \pi \int_0^4 \left(16y^2 - 8y^3 + y^4\right) dy$$

$$= \pi \left(\frac{16}{3}y^3 - 2y^4 + \frac{1}{5}y^5\right)\Big|_0^4 = \pi \left(\frac{1024}{3} - 512 + \frac{1024}{5}\right) = \frac{512\pi}{15}.$$

Solving for y as a function of x, $4 - x = 4 - 4y + y^2 = (y - 2)^2$ or $y = 2 \pm \sqrt{4 - x}$. Thus R is the region below $y = 2 + \sqrt{4 - x}$ above $y = 2 - \sqrt{4 - x}$ between $x = 0$ and $x = 4$. Thus shells gives

$$V = \int_0^4 2\pi x \left((2 + \sqrt{4 - x}) - (2 - \sqrt{4 - x})\right) dx = 4\pi \int_0^4 x\sqrt{4 - x}dx.$$

Substituting $u = 4 - x$, $du = -dx$, when $x = 0$, $u = 4$ and when $x = 4$, $u = 0$ thus

$$V = -4\pi \int_4^0 (4 - u)\sqrt{u}du = 4\pi \int_0^4 (4u^{1/2} - u^{3/2})du$$

$$= 4\pi \left(\frac{8}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right)\Big|_0^4 = 4\pi \left(\frac{64}{3} - \frac{64}{5}\right) = \frac{512\pi}{15}.$$

10. [10 points] Find the length of the curve $y = 2(x-2)^{3/2}$ from $x = 2$ to $x = 9$.

We compute $\frac{dy}{dx} = 3(x-2)^{1/2}$ hence

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + 9(x-2) = 9x - 17$$

and

$$L = \int_2^9 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_2^9 (9x-17)^{1/2} dx = \frac{2}{27}(9x-17)^{3/2}\Big|_2^9$$

$$= \frac{2}{27}((64)^{3/2} - 1^{3/2}) = \frac{1022}{27}.$$

11. [10 points] Express the area of the surface S obtained by revolving the curve $y = x^2$ for $0 \leq x \leq 3$ about the x-axis as a definite integral, but do not attempt to evaluate the integral.

Since $\frac{dy}{dx} = 2x$ we have

$$A = \int_0^3 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^3 2\pi x^2 \sqrt{1 + 4x^2} dx.$$

12. [15 points] The population of Houston in 1960 was 1 million people and in 2000 it was 2 million people. Assuming exponential growth, find the population of Houston as a function of time. What will be the population of Houston in 2050? When will the population of Houston be 3 million?

Let t be the time in years since 1960 and $P(t)$ the population in millions at time t. By exponential growth $P(t) = Ce^{kt}$ for some constants C and k. We have $P(0) = 1 = Ce^0 = C$ and hence $P(40) = 2 = e^{40k}$. Hence $\ln 2 = 40k$ or $k = \frac{1}{40} \ln 2$.

Thus $P(t) = \exp(\frac{t \ln 2}{40})$.

Hence the population in 2050 will be $P(90) = \exp(\frac{90 \ln 2}{40}) = \exp(\frac{9 \ln 2}{4}) = 2^{9/4}$ or about 4.75 million.

The population will be 3 million when $3 = P(t) = \exp(\frac{t \ln 2}{40})$ or $t \ln 2 = 40 \ln 3$ or $t = 40 \ln 3 / \ln 2$ or sometime in 2023.