Math 101 Fall 2002 Final Exam **Solutions**

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Friday, December 13, 2002

Instructions: This is a closed book, closed notes exam. Use of calculators is not permitted. You have **three hours**. Do all 12 problems. Please do all your work on the paper provided.
Please print you name clearly here.

Print name: ____

Upon finishing please sign the pledge below:
On my honor I have neither given nor received any aid on this exam.

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Grader's use only:

1. ________ /25

2. ________ /15

3. ________ /20

4. ________ /10

5. ________ /10

6. ________ /15

7. ________ /15

8. ________ /20

9. ________ /15

10. ________ /20

11. ________ /20

12. ________ /15

1. [25 points] Evaluate the derivatives of the following functions

(a) $g(x) = \frac{\sin x}{1+\tan x}$

$$\begin{aligned} g'(x) &= \frac{(1+\tan x)\frac{d(\sin x)}{dx} - \sin x \frac{d(1+\tan x)}{dx}}{(1+\tan x)^2} \\ &= \frac{(1+\tan x)\cos x - \sin x \sec^2 x}{(1+\tan x)^2}. \end{aligned}$$

(b) $h(t) = \sqrt{1+e^{-2t}}$

$$h'(t) = \frac{1}{2\sqrt{1+e^{-2t}}} \frac{d(1+e^{-2t})}{dt} = \frac{-2e^{-2t}}{2\sqrt{1+e^{-2t}}} = \frac{-e^{-2t}}{\sqrt{1+e^{-2t}}}.$$

(c) $F(x) = \arcsin(1-x^2)$

$$F'(x) = \frac{1}{\sqrt{1-(1-x^2)^2}} \frac{d(1-x^2)}{dx} = \frac{-2x}{\sqrt{1-(1-x^2)^2}}.$$

(d) $G(t) = (1+\sec(t^3-1))^2$

$$\begin{aligned} G'(t) &= 2(1+\sec(t^3-1))\frac{d(\sec(t^3-1))}{dt} = 2(1+\sec(t^3-1))\sec(t^3-1)\tan(t^3-1)\frac{d(t^3-1)}{dt} \\ &= 6t^2(1+\sec(t^3-1))\sec(t^3-1)\tan(t^3-1). \end{aligned}$$

(e) $H(x) = \int_2^{x^2} \frac{1}{\ln t} dt$

$$H'(x) = \frac{1}{\ln(x^2)} \frac{d(x^2)}{dx} = \frac{2x}{\ln(x^2)}.$$

2. [15 points] Evaluate the following limits, if they exist.

(a) $\lim_{x\to -2} \frac{x^3+x^2+4}{x^2+x-2}$

This limit is indeterminate of type 0/0 so by L'Hôpital

$$\lim_{x\to -2} \frac{x^3+x^2+4}{x^2+x-2} = \lim_{x\to -2} \frac{3x^2+2x}{2x+1} = \frac{12-4}{-4+1} = -\frac{8}{3}.$$

(b) $\lim_{x\to 0}(e^{2x}-1)\cot x$

This limit is indeterminate of type $0\cdot\infty$ so rearranging and applying L'Hôpital give

$$\lim_{x\to 0}(e^{2x}-1)\cot x = \lim_{x\to 0} \frac{e^{2x}-1}{\tan x} = \lim_{x\to 0} \frac{2e^{2x}}{\sec^2 x} = \frac{2}{1} = 2.$$

(c) $\lim_{x\to\infty} \left(1-\frac{2}{x}\right)^{x+2}$

This limit is indeterminate of type 1^∞ so rearranging and applying L'Hôpital gives

$$\begin{aligned}
\lim_{x\to\infty} \left(1-\frac{2}{x}\right)^{x+2} &= \lim_{x\to\infty} \exp\left((x+2)\ln\left(1-\frac{2}{x}\right)\right) \\
&= \exp\left(\lim_{x\to\infty} \frac{\ln\left(1-\frac{2}{x}\right)}{1/(x+2)}\right) \\
&= \exp\left(\lim_{x\to\infty} \frac{\frac{2/x^2}{(1-\frac{2}{x}}}{-1/(x+2)^2}\right) \\
&= \exp\left(\lim_{x\to\infty} -\frac{2(x+2)^2}{x^2-2x}\right) = \exp\left(\lim_{x\to\infty} -\frac{4(x+2)}{2x-2}\right) \\
&= \exp\left(\lim_{x\to\infty} -\frac{4}{2}\right) = \exp(-2) = e^{-2}.
\end{aligned}$$

3. [20 points] Evaluate the following integrals

(a) $\int 3xe^{x^2+1}dx$

Substituting $u = x^2 + 1$, $du = 2x\,dx$ gives

$$\int 3xe^{x^2+1}dx = \frac{3}{2}\int e^u\,du = \frac{3}{2}e^u + C = \frac{3}{2}e^{x^2+1} + C.$$

(b) $\int_0^\pi \frac{\sin x}{3-2\cos x}dx$

Substituting $u = 3-2\cos x$, $du = 2\sin x\,dx$, $x = 0$ means $u = 1$ and $x = \pi$ means $u = 5$ gives

$$\int_0^\pi \frac{\sin x}{3-2\cos x}dx = \frac{1}{2}\int_1^5 \frac{du}{u} = \frac{1}{2}\ln u|_1^5 = \frac{1}{2}(\ln 5 - \ln 1) = \frac{1}{2}\ln 5.$$

(c) $\int \frac{x^2}{1+x^6}dx$

Substituting $u = x^3$, $du = 3x^2\,dx$ gives

$$\int \frac{x^2}{1+x^6}dx = \frac{1}{3}\int \frac{du}{1+u^2} = \frac{1}{3}\arctan u + C = \frac{1}{3}\arctan(x^3) + C.$$

(d) $\int (1+\tan x)^3\sec^2 x dx$

Substituting $u = 1+\tan x$, $du = \sec^2 x\,dx$ gives

$$\int (1+\tan x)^3\sec^2 x dx = \int u^3 du = \frac{1}{4}u^4 + C = \frac{1}{4}(1+\tan x)^4 + C.$$

4. [10 points] Find the area of the region in the plane bounded by the curves $y = x - x^2$ and $y = 2x - 2$.

The two curves meet when $x - x^2 = 2x - 2$ or $0 = x^2 + x - 2 = (x-1)(x+2)$, hence at $x = 1$ and $x = -2$. At $x = 0$, the parabola has $y = 0$ and the line has $y = -2$. Hence the parabola is the higher curve for $-2 \leq x \leq 1$. Hence the area is

$$\begin{aligned} A &= \int_{-2}^{1} (x - x^2 - (2x - 2))\, dx = \int_{-2}^{1} (2 - x - x^2)\, dx \\ &= \left(2x - \frac{1}{2}x^2 - \frac{1}{3}x^3\right)\Big|_{-2}^{1} = \left(2 - \frac{1}{2} - \frac{1}{3}\right) - \left(-4 - 2 + \frac{8}{3}\right) \\ &= \frac{9}{2}. \end{aligned}$$

5. [10 points] Find the equation of the tangent line to the function $y = \ln(1 + 2x)$ at $x = 0$.

Since

$$\frac{d}{dx} \ln(1 + 2x) = \frac{2}{1 + 2x}$$

the slope of the tangent line is

$$m = \frac{d}{dx} \ln(1 + 2x)\Big|_{x=0} = \frac{2}{1 + 0} = 2.$$

Since the tangent line goes through $(0, \ln(1 + 0)) = (0, 0)$, the equation is $(y - 0) = 2(x - 0)$ or $y = 2x$.

6. [15 points] Let R be the region in the plane bounded by the curve $y = \sqrt{x} - 2$, the line $x = 9$ and the x-axis. Let S be the solid that results from revolving R about the x-axis. Express the volume of S as a definite integral in TWO ways, using the method of washers and the method of shells. Evaluate ONE of these two integrals (your choice).

Since the region is revolved about the x-axis, to use the method of washers we must do an integral with repsect to x. The smallest x value that occurs is $x = 4$ where the parabola $y = \sqrt{x} - 2$ crosses the x-axis. The largest is $x = 9$ at the vertical line. For $4 \leq x \leq 9$ the parabola is over the x-axis, so the volume is

$$\begin{aligned} V &= \int_4^9 \pi \left((\sqrt{x} - 2)^2 - 0^2 \right) \, dx = \pi \int_4^9 (x - 4\sqrt{x} + 4) \, dx \\ &= \pi \left(\frac{1}{2}x^2 - \frac{8}{3}x^{3/2} + 4x \right) \Big|_4^9 \\ &= \pi \left(\left(\frac{81}{2} - 72 + 36 \right) - \left(8 - \frac{64}{3} + 16 \right) \right) \\ &= \frac{11\pi}{6}. \end{aligned}$$

To use the method of shells we must do an integral with respect to y. The smallest y value is $y = 0$ at the x-axis and the highest is $y = 1$ when the parabola $y = \sqrt{x} - 2$ crosses the vertical line $x = 9$. For $0 \leq y \leq 1$ the right side of the region is the vertical line $x = 9$ and the left side is $y = \sqrt{x} - 2$ or $x = (y = 2)^2$. Hence the volume is

$$\begin{aligned} V &= \int_0^1 2\pi y (9 - (y + 2)^2) \, dy = 2\pi \int_0^1 (5y - 4y^2 - y^3) \, dy \\ &= 2\pi \left(\frac{5}{2}y^2 - \frac{4}{3}y^3 - \frac{1}{4}y^4 \right) \Big|_0^1 \\ &= 2\pi \left(\frac{5}{2} - \frac{4}{3} - \frac{1}{4} \right) = \frac{11\pi}{6}. \end{aligned}$$

7. [15 points] Consider the curve C given by $y = (e^x + e^{-x})/2$ for $0 \le x \le 2$.

(a) Find the length of the curve C.

Let $f(x) = (e^x + e^{-x})/2$, then $f'(x) = (e^x - e^{-x})/2$ so

$$(f'(x))^2 = \frac{1}{4}e^{2x} - \frac{1}{2} + \frac{1}{4}e^{-2x}$$

and

$$1 + (f'(x))^2 = \frac{1}{4}e^{2x} + \frac{1}{2} + \frac{1}{4}e^{-2x} = \left(\frac{e^x + e^{-x}}{2}\right)^2.$$

Hence the arc length is

$$L = \int_0^2 \left(\frac{e^x + e^{-x}}{2}\right) dx = \left(\frac{e^x - e^{-x}}{2}\right)\Bigg|_0^2 = \frac{e^2 - e^{-2}}{2}.$$

(b) Express the area of the surface that results from revolving C about the y-axis as a definite integral, but do not attempt to evaluate the integral.

Since the curve is revolved about the y-axis the radius is x so the surface area is

$$S = \int_0^2 2\pi x \left(\frac{e^x + e^{-x}}{2}\right) dx.$$

8. [20 points] For the function $f(x) = \frac{1}{5}x^5 + \frac{1}{4}x^4 - \frac{8}{3}x^3 - 6x^2$, the first two derivatives are $f'(x) = x^4 + x^3 - 8x^2 - 12x = x(x+2)^2(x-3)$ and $f''(x) = 4x^3 + 3x^2 - 16x - 12 = (4x+3)(x+2)(x-2)$. YOU ARE NOT REQUIRED TO VERIFY THESE FORMULAS. For all other aspects of this problem you are required to justify your answer.

(a) Find the open intervals on which the function f is increasing and those on which it is decreasing.

x is positive for $x > 0$ and negative for $x < 0$. $(x+2)^2$ is always non-negative but it zero at $x = -2$. $x - 3$ is positive for $x > 3$ and negative for $x < 3$. Hence $f'(x) = x(x+2)^2(x-3)$ is positive for $x < -2$ and for $-2 < x < 0$, negative for $0 < x < 3$, and positive for $x > 3$. Thus f is increasing on $(-\infty, -2)$ and on $(-2, 0)$, f is decreasing on $(0, 3)$ and f is increasing on $(3, \infty)$. Since $f(x)$ is continuous at $x = -2$, it is also correct to say f is increasing on $(-\infty, 0)$.

(b) Find all critical points of $f(x)$ and classify them as local maxima, local minima or neither.

f' always exists so the critical points are where $f'(x) = 0$, that is at $x = -2$, $x = 0$ and $x = 3$. From the first derivative test and part (a), $x = -2$ is neither a local maximum nor a local minimum, $x = 0$ is a local maximum and $x = 3$ is a local minimum. The last two critical points can also be classified using the second derivative test since $f''(0) = -12 < 0$ and $f''(3) = 75 > 0$.

(c) Find the intervals on which the function f is concave upward and concave downward.

$4x + 3$ is positive for $x > -3/4$ and negative for $x < -3/4$. $x + 2$ is positive for $x > -2$ and negative for $x < -2$. $x - 2$ is positive for $x > 2$ and negative for $x < 2$. Hence $f''(x) = (4x+3)(x+2)(x-2)$ is negative for $x < -2$, positive for $-2 < x < -3/4$, negative for $-3/4 < x < 2$ and positive for $x > 2$. Hence f is concave upward of $(-2, -3/4)$ and on $(2, \infty)$ and f is concave downward on $(-\infty, -2)$ and on $(-3/4, 2)$.

9. [15 points] When salt is dissolved in water, the amount A of salt that remains undissolved after t minutes satisfies the differential equation $\frac{dA}{dt} = -kA$, $(k > 0)$. If 25% dissolves after 2 minutes, how long does it take for half the salt to dissolve? NOTE: you do not have to simplify your final answer.

Since the rate of change of A is proportional to A, we have $A(t) = A_0 e^{-kt}$ for some constants $A_0 = A(0)$ and $k > 0$. Since 25% has dissolved at $t = 2$, we have $A(2) = 3A_0/4$ and plugging in $t = 2$ gives

$$3A_0/4 = A(2) = A_0 e^{-2k} \text{ or } e^{2k} = 4/3 \text{ or } k = \frac{1}{2}\ln(4/3).$$

The time t at which half dissolves satisfies $A(t) = A_0/2 = A_0 e^{-kt}$ or $e^{kt} = 2$ or

$$t = \frac{\ln 2}{k} = \frac{2\ln 2}{\ln(4/3)}.$$

10. [20 points] We wish to make an ice cream cone with a volume of 18π in^3. What dimensions give the minimum possible area? You must justify that your answer really is the global minimum. (The area of a cone of height h and radius r is $A = \pi r\sqrt{r^2+h^2}$ and the volume is $V = \frac{1}{3}\pi r^2 h$.)

Let r be the radius of the cone and h the height in inches. Since the volume is 18π these are related by

$$18\pi = \frac{1}{3}\pi r^2 h \text{ or } r^2 = \frac{54}{h}.$$

Rather than minimizing the area, it is easier to minimize the squared area (SA) which is

$$SA(h) = \pi^2 r^2(r^2+h^2) = \pi^2\frac{54}{h}\left(\frac{54}{h}+h^2\right) = 54\pi^2\left(\frac{54}{h^2}+h\right).$$

Since h and r are lengths we must have $h > 0$ hence the domain is $(0,\infty)$. We compute

$$SA'(h) = 54\pi^2\left(-\frac{108}{h^3}+1\right) = \frac{54\pi^2(h^3-108)}{h^3}.$$

The only critical point is where this is zero at $h = (108)^{1/3}$. Further $SA'(h)$ is negative for $h < (108)^{1/3}$ and positive for $h > (108)^{1/3}$, hence from the first derivative test $h = (108)^{1/3}$ is a global minimum. Thus the dimensions of the minimum are $h = (108)^{1/3}$ and $r = \sqrt{54/(108)^{1/3}}$.

The fact that the critical point is a global minimum can also be proved using the second derivative test by noting that

$$SA''(h) = 54\pi^2\frac{324}{h^4}$$

is positive for ALL $h > 0$.

11. [20 points] The volume of a cone is $\frac{1}{3}\pi r^2 h$ where r is the radius and h is the height. Suppose the radius and height of a cone are changing with time. In particular the radius is increasing at a constant rate of 2 cm/s while the height is decreasing at a constant rate of 3 cm/s. When the radius is 4cm and the height is 6cm at what rate is the volume changing?

Since $V = \frac{1}{3}\pi r^2 h$ holds for all time, we have

$$\frac{dV}{dt} = \frac{1}{3}\pi \left(2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\right).$$

Hence when $r = 4$, $h = 6$, $\frac{dr}{dt} = 2$ and $\frac{dh}{dt} = -3$ we have

$$\frac{dV}{dt} = \frac{1}{3}\pi \left(2 \cdot 4 \cdot 6 \cdot 2 + 4^2 \cdot (-3)\right) = 16\pi.$$

So the volume is increasing at 16π cm^3/sec.

12. [15 points] Compute the definite integral $\int_0^2 (1+3x^2)dx$ from the definition, that is, by finding $\lim_{n\to\infty} \sum_{i=1}^n f(x_i)\Delta x$. No credit will be given for computing the integral in any other way.

The integral is from $a = 0$ to $b = 2$ of $f(x) = 1 + 3x^2$. We compute $\Delta x = (b-a)/n = 2/n$ and $x_i = a + i\Delta x = 0 + i(2/n) = 2i/n$. Hence

$$\begin{aligned}
\sum_{i=1}^n f(x_i)\Delta x &= \sum_{i=1}^n \left(1 + 3x_i^2\right)\Delta x = \sum_{i=1}^n \left(1 + 3\left(\frac{2i}{n}\right)^2\right)\frac{2}{n} \\
&= \sum_{i=1}^n \left(\frac{2}{n} + \frac{24i^2}{n^3}\right) = \frac{2}{n}\left(\sum_{i=1}^n 1\right) + \frac{24}{n^3}\left(\sum_{i=1}^n i^2\right) \\
&= \frac{2}{n}\cdot n + \frac{24}{n^3}\left(\frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n\right) = 2 + 8 + \frac{12}{n} + \frac{4}{n^2} \\
&= 10 + \frac{12}{n} + \frac{4}{n^2}.
\end{aligned}$$

Hence

$$\int_0^2 \left(1 + 3x^2\right)dx = \lim_{n\to\infty}\sum_{i=1}^n f(x_i)\Delta x = \lim_{n\to\infty}\left(10 + \frac{12}{n} + \frac{4}{n^2}\right) = 10.$$